Zeitschrift: L'Enseignement Mathématique

Herausgeber: Commission Internationale de l'Enseignement Mathématique

Band: 44 (1998)

Heft: 1-2: L'ENSEIGNEMENT MATHÉMATIQUE

Artikel: THREE DISTANCE THEOREMS AND COMBINATORICS ON WORDS

Autor: Alessandri, Pascal / Berthé, Valérie

Kapitel: 2.3 Frequencies of factors of Sturmian sequences

DOI: https://doi.org/10.5169/seals-63900

REMARK. In fact, one can prove the following: the frequencies of factors of length n take at most $p(n+1)-p(n)+r_n+l_n$ values, where r_n (respectively, l_n) denotes the number of factors having more than one right (respectively, left) extension.

We deduce from this theorem that if $p(n+1)-p(n)$ is uniformly bounded with n, the frequencies of factors of given length take a finite number of values. Indeed, using a theorem of Cassaigne quoted below (see [10]), we can easily state the following corollary.

THEOREM 7. *If the complexity $p(n)$ of a sequence with values in a finite alphabet is sub-affine then $p(n+1)-p(n)$ is bounded.*

COROLLARY 1. *If a sequence over a finite alphabet has a sub-affine complexity, then the frequencies of its factors of given length take a finite number of values.*

Examples of sequences with sub-affine complexity function include the fixed point of a uniform substitution (i.e., of a substitution such that the images of the letters have the same length), the coding of a rotation or the coding of the orbit of a point under an interval exchange map with respect to the intervals of the interval exchange map.

2.3 FREQUENCIES OF FACTORS OF STURMIAN SEQUENCES

In particular, in the Sturmian case ($p(n) = n+1$, for every integer n), Theorem 6 implies the following result (see [3]).

THEOREM 8. *The frequencies of factors of given length of a Sturmian sequence take at most three values.*

Consider a Sturmian sequence of angle α. We have seen in Section 2.1 that the frequency of a factor $w_1 \cdots w_n$ of u is equal to the length of the interval

$$I(w_1, \ldots, w_n) = \bigcap_{j=0}^{n-1} R^{-j}(I_{w_{j+1}}),$$

and that these sets $I(w_1, \ldots, w_n)$ are exactly the intervals bounded by the points

$$0, \{1-\alpha\}, \ldots, \{n(1-\alpha)\}.$$

We deduce from Theorem 8 that the lengths of the intervals $I(w_1, \dots, w_n)$, and thus the lengths of the intervals obtained by placing the points $0, \{1-\alpha\}, \dots, \{n(1-\alpha)\}$ on the unit circle, take at most three values. Hence Theorem 8 is equivalent to the three distance theorem and provides a combinatorial proof of this result.

REMARK. In fact this point of view, and more precisely the study of the evolution of the graphs of words with respect to the length n of the factors, allows us to give a proof of the most complete version of the three distance theorem as given in [53] (for more details, the reader is referred to [3]).

3. THE THREE DISTANCE THEOREM

The three distance theorem was initially conjectured by Steinhaus, first proved by V. T. Sós (see [53] and also [52]), and then by Świerczkowski [56], Surányi [55], Slater [51], Halton [31]. More recent proofs have also been given by van Ravenstein [44] and Langevin [35]. A survey of the different approaches used by these authors is to be found in [44, 51, 35]. In the literature this theorem is called *the Steinhaus theorem, the three length, three gap* or *the three step theorem*. In order to avoid any ambiguity, we will always call it the three distance theorem, reserving the name *three gap* for the theorem introduced in the next section.

THREE DISTANCE THEOREM. *Let $0 < \alpha < 1$ be an irrational number and n a positive integer. The points $\{i\alpha\}$, for $0 \le i \le n$, partition the unit circle into $n+1$ intervals, the lengths of which take at most three values, one being the sum of the other two.*

More precisely, let $\left(\frac{p_k}{q_k}\right)_{k\in\mathbf{N}}$ and $(c_k)_{k\in\mathbf{N}}$ be the sequences of convergents and partial quotients associated to α in its continued fraction expansion (if $\alpha = [0, c_1, c_2, \dots]$, then $\frac{p_n}{q_n} = [0, c_1, \dots, c_n]$). Let $\eta_k = (-1)^k(q_k\alpha - p_k)$. Let n be a positive integer. There exists a unique expression for n of the form

$$n = mq_k + q_{k-1} + r\,,$$

with $1 \le m \le c_{k+1}$ and $0 \le r < q_k$. Then the circle is divided by the points $0, \{\alpha\}, \{2\alpha\}, \dots, \{n\alpha\}$ into $n+1$ intervals which satisfy: