

**Zeitschrift:** L'Enseignement Mathématique  
**Herausgeber:** Commission Internationale de l'Enseignement Mathématique  
**Band:** 44 (1998)  
**Heft:** 1-2: L'ENSEIGNEMENT MATHÉMATIQUE

**Artikel:** THREE DISTANCE THEOREMS AND COMBINATORICS ON WORDS  
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**Kapitel:** 2.3 Frequencies of factors of Sturmian sequences  
**DOI:** <https://doi.org/10.5169/seals-63900>

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REMARK. In fact, one can prove the following: the frequencies of factors of length  $n$  take at most  $p(n+1) - p(n) + r_n + l_n$  values, where  $r_n$  (respectively,  $l_n$ ) denotes the number of factors having more than one right (respectively, left) extension.

We deduce from this theorem that if  $p(n+1) - p(n)$  is uniformly bounded with  $n$ , the frequencies of factors of given length take a finite number of values. Indeed, using a theorem of Cassaigne quoted below (see [10]), we can easily state the following corollary.

THEOREM 7. *If the complexity  $p(n)$  of a sequence with values in a finite alphabet is sub-affine then  $p(n+1) - p(n)$  is bounded.*

COROLLARY 1. *If a sequence over a finite alphabet has a sub-affine complexity, then the frequencies of its factors of given length take a finite number of values.*

Examples of sequences with sub-affine complexity function include the fixed point of a uniform substitution (i.e., of a substitution such that the images of the letters have the same length), the coding of a rotation or the coding of the orbit of a point under an interval exchange map with respect to the intervals of the interval exchange map.

### 2.3 FREQUENCIES OF FACTORS OF STURMIAN SEQUENCES

In particular, in the Sturmian case ( $p(n) = n + 1$ , for every integer  $n$ ), Theorem 6 implies the following result (see [3]).

THEOREM 8. *The frequencies of factors of given length of a Sturmian sequence take at most three values.*

Consider a Sturmian sequence of angle  $\alpha$ . We have seen in Section 2.1 that the frequency of a factor  $w_1 \cdots w_n$  of  $u$  is equal to the length of the interval

$$I(w_1, \dots, w_n) = \bigcap_{j=0}^{n-1} R^{-j}(I_{w_{j+1}}),$$

and that these sets  $I(w_1, \dots, w_n)$  are exactly the intervals bounded by the points

$$0, \{1 - \alpha\}, \dots, \{n(1 - \alpha)\}.$$

We deduce from Theorem 8 that the lengths of the intervals  $I(w_1, \dots, w_n)$ , and thus the lengths of the intervals obtained by placing the points  $0, \{1 - \alpha\}, \dots, \{n(1 - \alpha)\}$  on the unit circle, take at most three values. Hence Theorem 8 is equivalent to the three distance theorem and provides a combinatorial proof of this result.

REMARK. In fact this point of view, and more precisely the study of the evolution of the graphs of words with respect to the length  $n$  of the factors, allows us to give a proof of the most complete version of the three distance theorem as given in [53] (for more details, the reader is referred to [3]).

### 3. THE THREE DISTANCE THEOREM

The three distance theorem was initially conjectured by Steinhaus, first proved by V. T. Sós (see [53] and also [52]), and then by Świerczkowski [56], Surányi [55], Slater [51], Halton [31]. More recent proofs have also been given by van Ravenstein [44] and Langevin [35]. A survey of the different approaches used by these authors is to be found in [44, 51, 35]. In the literature this theorem is called *the Steinhaus theorem*, *the three length, three gap* or *the three step theorem*. In order to avoid any ambiguity, we will always call it the three distance theorem, reserving the name *three gap* for the theorem introduced in the next section.

THREE DISTANCE THEOREM. *Let  $0 < \alpha < 1$  be an irrational number and  $n$  a positive integer. The points  $\{i\alpha\}$ , for  $0 \leq i \leq n$ , partition the unit circle into  $n + 1$  intervals, the lengths of which take at most three values, one being the sum of the other two.*

*More precisely, let  $(\frac{p_k}{q_k})_{k \in \mathbb{N}}$  and  $(c_k)_{k \in \mathbb{N}}$  be the sequences of convergents and partial quotients associated to  $\alpha$  in its continued fraction expansion (if  $\alpha = [0, c_1, c_2, \dots]$ , then  $\frac{p_n}{q_n} = [0, c_1, \dots, c_n]$ ). Let  $\eta_k = (-1)^k(q_k\alpha - p_k)$ . Let  $n$  be a positive integer. There exists a unique expression for  $n$  of the form*

$$n = mq_k + q_{k-1} + r,$$

*with  $1 \leq m \leq c_{k+1}$  and  $0 \leq r < q_k$ . Then the circle is divided by the points  $0, \{\alpha\}, \{2\alpha\}, \dots, \{n\alpha\}$  into  $n + 1$  intervals which satisfy:*