

Zeitschrift:	L'Enseignement Mathématique
Herausgeber:	Commission Internationale de l'Enseignement Mathématique
Band:	44 (1998)
Heft:	1-2: L'ENSEIGNEMENT MATHÉMATIQUE
Artikel:	ON CONNES' JOINT DISTRIBUTION TRICK AND A NOTION OF AMENABILITY FOR POSITIVE MAPS
Autor:	POPA, Sorin
Kapitel:	1. Proof of the theorem
DOI:	https://doi.org/10.5169/seals-63896

Nutzungsbedingungen

Die ETH-Bibliothek ist die Anbieterin der digitalisierten Zeitschriften auf E-Periodica. Sie besitzt keine Urheberrechte an den Zeitschriften und ist nicht verantwortlich für deren Inhalte. Die Rechte liegen in der Regel bei den Herausgebern beziehungsweise den externen Rechteinhabern. Das Veröffentlichen von Bildern in Print- und Online-Publikationen sowie auf Social Media-Kanälen oder Webseiten ist nur mit vorheriger Genehmigung der Rechteinhaber erlaubt. [Mehr erfahren](#)

Conditions d'utilisation

L'ETH Library est le fournisseur des revues numérisées. Elle ne détient aucun droit d'auteur sur les revues et n'est pas responsable de leur contenu. En règle générale, les droits sont détenus par les éditeurs ou les détenteurs de droits externes. La reproduction d'images dans des publications imprimées ou en ligne ainsi que sur des canaux de médias sociaux ou des sites web n'est autorisée qu'avec l'accord préalable des détenteurs des droits. [En savoir plus](#)

Terms of use

The ETH Library is the provider of the digitised journals. It does not own any copyrights to the journals and is not responsible for their content. The rights usually lie with the publishers or the external rights holders. Publishing images in print and online publications, as well as on social media channels or websites, is only permitted with the prior consent of the rights holders. [Find out more](#)

Download PDF: 07.08.2025

ETH-Bibliothek Zürich, E-Periodica, <https://www.e-periodica.ch>

1. PROOF OF THE THEOREM

Let $X = \mathbf{R}_+^2 \setminus \{0\}$ and $H_0(x, y) = x$, $H_i(x, y) = y$, $i = 1, 2, \dots, n$. As in ([C], page 77), it follows that

$$\mu_i(A_0 \times A_i) \stackrel{\text{def}}{=} \text{Tr}(\Phi_0(e_{A_0}(b_0))\Phi_i(e_{A_i}(b_i))),$$

for $A_j \subset \mathbf{R}_+$, $0 \leq j \leq n$, Borel sets such that for each $i \geq 1$ either $0 \notin \bar{A}_0$ or $0 \notin \bar{A}_i$, defines a Radon measure μ_i on X , which satisfies the properties:

- (a) $\|f(H_i)\|_{1, \mu_i} = \text{Tr}(\Phi_i(|f|(b_i)))$ (resp., $\|f(H_i)\|_{2, \mu_i}^2 = \text{Tr}(\Phi_i(|f|^2(b_i))) \leq \|f(b_i)\|_{2, \text{Tr}}^2$) for all Borel functions $f: [0, \infty) \rightarrow \mathbf{C}$ with $f(0) = 0$ and $f(b_i) \in L^1(P_1, \text{Tr})$ (respectively $f(b_i) \in L^2(P_1, \text{Tr})$), $i = 0, 1, \dots, n$.
- (b) $\int_X f_0(H_0) \overline{f_i(H_i)} d\mu_i = \text{Tr}(\Phi_0(f_0(b_0))\Phi_i(\bar{f}_i(b_i)))$, for all $f_i: [0, \infty) \rightarrow \mathbf{C}$ Borel with $f_i(0) = 0$ and $f_i(b_i) \in L^2(P_1, \text{Tr})$, $\forall i = 0, 1, \dots, n$.
- (c) $\|f_0(H_0) - f_i(H_i)\|_{2, \mu} \geq \|\Phi_0(f_0(b_0)) - \Phi_i(f_i(b_i))\|_{2, \text{Tr}}$, for all f_i as in (b).
- (d) $\|H_0 - H_i\|_{2, \mu_i}^2 = \text{Tr}(\Phi_0(b_0^2)) + \text{Tr}(\Phi_i(b_i^2)) - 2 \text{Tr}(\Phi_0(b_0)\Phi_i(b_i)) \leq 6\delta$.

Proof of (a)–(d). Indeed, (a) and (b) are clear by the proof of I.1 in [C] and the definition of μ_i . Further on, by (a), (b), (1), and Kadison's inequality (which asserts that positive, linear, unital maps φ between C^* algebras satisfy $\varphi(b)\varphi(b) \leq \varphi(b^2)$ for any $b = b^*$), we get:

$$\begin{aligned} \|f_0(H_0) - f_i(H_i)\|_{2, \mu_i}^2 &= \|f_0(H_0)\|_{2, \mu_i}^2 + \|f_i(H_i)\|_{2, \mu}^2 - 2 \operatorname{Re} \int_X f_0(H_0) \overline{f_i(H_i)} d\mu \\ &= \text{Tr}(\Phi_0(f_0(b_0))^* f_0(b_0)) \\ &\quad + \text{Tr}(\Phi_i(f_i(b_i))^* f_i(b_i)) - 2 \operatorname{Re} \text{Tr}(\Phi_0(f_0(b_0))\Phi_i(\bar{f}_i(b_i))) \\ &\geq \text{Tr}(\Phi_0(f_0(b_0))^* \Phi_0(f_0(b_0))) \\ &\quad + \text{Tr}(\Phi_i(f_i(b_i))^* \Phi_i(f_i(b_i))) - 2 \operatorname{Re} \text{Tr}(\Phi_0(f_0(b_0))\Phi_i(f_i(b_i))^*) \\ &= \|\Phi_0(f_0(b_0)) - \Phi_i(f_i(b_i))\|_{2, \text{Tr}}^2. \end{aligned}$$

This proves (c). Then (d) is clear by noticing that the hypothesis and the Cauchy-Schwarz inequality imply:

$$\begin{aligned} \text{Tr}(\Phi_0(b_0^2)) + \text{Tr}(\Phi_i(b_i^2)) - 2 \text{Tr}(\Phi_0(b_0)\Phi_i(b_i)) \\ &\leq \text{Tr}(b_0^2) + \text{Tr}(b_i^2) - 2 \text{Tr}(\Phi_0(b_0)\Phi_i(b_i)) \\ &= 2 - 2 \text{Tr}(\Phi_0(b_0)\Phi_i(b_i)) \\ &\leq 2 - 2 \text{Tr}(\Phi_0(b_0)^2) + 2\delta \\ &\leq 2(1 - (1 - \delta)^2) + 2\delta \leq 6\delta, \end{aligned}$$

thus ending the proof of properties (a)–(d). \square

Proof of (i) in the Theorem. To prove (i), remark that we have, like in proof of 1.2.6 in [C], the estimate:

$$\begin{aligned} \int_{\mathbf{R}_+^*} \|e_{t^{1/2}}(H_0) - e_{t^{1/2}}(H_i)\|_{2,\mu_i}^2 dt \\ = \|H_0^2 - H_i^2\|_{1,\mu_i} \leq \|H_0 - H_i\|_{2,\mu_i} \|H_0 + H_i\|_{2,\mu_i}. \end{aligned}$$

But (d) implies $\|H_0 - H_i\|_{2,\mu_i} \leq (6\delta)^{1/2}$ and (a) implies $\|H_0 + H_i\|_{2,\mu_i} \leq \|H_0\|_{2,\mu_i} + \|H_i\|_{2,\mu_i} \leq \|b_0\|_{2,\text{Tr}} + \|b_i\|_{2,\text{Tr}} = 2$. Thus, by applying (c) to the functions $f_i = \chi_{[t^{1/2}, \infty)}$, $0 \leq i \leq n$, for each $t > 0$, and summing up the above inequalities over i we obtain

$$\begin{aligned} (*) \quad \int_{\mathbf{R}_+^*} \sum_{i=1}^n \|\Phi_0(e_{t^{1/2}}(b_0)) - \Phi_i(e_{t^{1/2}}(b_i))\|_{2,\text{Tr}}^2 dt \\ \leq 2n(6\delta)^{1/2} = 2n(6\delta)^{1/2} \int_{\mathbf{R}_+^*} \|e_{t^{1/2}}(b_0)\|_{2,\text{Tr}}^2 dt. \end{aligned}$$

This implies that if we denote by D the set of all $t > 0$ for which

$$g(t) \stackrel{\text{def}}{=} \sum_{i=1}^n \|\Phi_0(e_{t^{1/2}}(b_0)) - \Phi_i(e_{t^{1/2}}(b_i))\|_{2,\text{Tr}}^2 dt < \delta^{1/4} \|e_{t^{1/2}}(b_0)\|_{2,\text{Tr}}^2,$$

then

$$\int_D \|e_{t^{1/2}}(b_0)\|_{2,\text{Tr}}^2 dt \geq 1 - 5n\delta^{1/4}.$$

Indeed, from $\int_D \|e_{t^{1/2}}(b_0)\|_{2,\text{Tr}}^2 dt < 1 - 5n\delta^{1/4}$, by taking into account that $g(t) \geq \delta^{1/4} \|e_{t^{1/2}}(b_0)\|_{2,\text{Tr}}^2$ for $t \in \mathbf{R}_+^* \setminus D$, we would get:

$$\begin{aligned} \int_{\mathbf{R}_+^*} g(t) dt &\geq \int_{\mathbf{R}_+^* \setminus D} g(t) dt \\ &\geq \delta^{1/4} \int_{\mathbf{R}_+^* \setminus D} \|e_{t^{1/2}}(b_0)\|_{2,\text{Tr}}^2 dt \\ &\geq 5n\delta^{1/2} > 2n(6\delta)^{1/2}. \end{aligned}$$

which is in contradiction with (*).

In particular, since $\delta < (5n)^{-4}$, we have $1 - 5n\delta^{1/4} > 0$ so that $D \neq \emptyset$. Thus, any $s > 0$ with $s^2 \in D$ will satisfy (i).

Proof of (ii) in the Theorem. To prove (ii), note first that $\text{Tr} \circ \Phi_0 \leq \text{Tr}$ already implies that for each fixed $x \in P_{2+}$ the map $L^1(P_1, \text{Tr}) \ni x_1 \mapsto \text{Tr}(x\Phi_0(x_1))$ defines a positive functional on $L^1(P_1, \text{Tr})$, which we denote

by $\Phi_0^*(x)$. Also, if we identify $L^1(P_1, \text{Tr})^*$ with P_1 , then $0 \leq x \leq 1$ implies $0 \leq \Phi_0^*(x) \leq 1$. Moreover, if in addition we have $\text{Tr} \circ \Phi_0 = \text{Tr}$, then $\Phi_0^*(1) = 1$, so Φ_0^* defines a positive, unital, linear mapping from P_2 into $P_1 = L^1(P_1, \text{Tr})^*$ satisfying $\text{Tr} \circ \Phi_0^* = \text{Tr}$. Consequently, if we denote $\Phi'_i = \Phi_0^* \circ \Phi_i: P_1 \rightarrow P_1$, $1 \leq i \leq n$, then $\Phi'_i(1) = 1$, $\text{Tr} \circ \Phi'_i \leq \text{Tr}$, $\forall i$, $1 \leq i \leq n$, and we have the estimates:

$$\begin{aligned} \|\Phi'_i(b_i) - b_0\|_{2,\text{Tr}}^2 &= \|\Phi'_i(b_i)\|_{2,\text{Tr}}^2 + \|b_0\|_{2,\text{Tr}}^2 - 2 \text{Tr}(\Phi_0^*(\Phi_i(b_i))b_0) \\ &\leq 2 - 2 \text{Tr}(\Phi_i(b_i)\Phi_0(b_0)) \\ &= 2 + \|\Phi_0(b_0) - \Phi_i(b_i)\|_{2,\text{Tr}}^2 - \|\Phi_0(b_0)\|_{2,\text{Tr}}^2 - \|\Phi_i(b_i)\|_{2,\text{Tr}}^2 \\ &\leq 2 + \delta^2 - 2(1 - \delta)^2 < 2\delta. \end{aligned}$$

In particular, if we denote $\delta' = (2\delta)^{1/2}$ then the above implies:

$$\|\Phi'_i(b_i)\|_{2,\text{Tr}} \geq \|b_i\|_{2,\text{Tr}} - (2\delta)^{1/2} = 1 - \delta'.$$

Altogether, this shows that we can apply the first part of the proof, with $\Phi'_0 = \text{id}, \Phi'_1, \dots, \Phi'_n$ instead of $\Phi_0, \Phi_1, \dots, \Phi_n$ and δ' instead of δ , with the same b_0, b_1, \dots, b_n , to obtain that the set D' of all $t > 0$ for which

$$\sum_{i=1}^n \|e_{t^{1/2}}(b_0) - \Phi'_i(e_{t^{1/2}}(b_i))\|_{2,\text{Tr}}^2 < \delta'^{1/4} \|e_{t^{1/2}}(b_0)\|_{2,\text{Tr}}^2$$

satisfies

$$\int_{D'} \|e_{t^{1/2}}(b_0)\|_{2,\text{Tr}}^2 dt \geq 1 - 5n\delta'^{1/4}.$$

Note that for $t \in D'$ we have:

$$\|e_{t^{1/2}}(b_0) - \Phi'_i(e_{t^{1/2}}(b_i))\|_{2,\text{Tr}} < \delta'^{1/8} \|e_{t^{1/2}}(b_0)\|_{2,\text{Tr}}$$

for all $i = 1, \dots, n$. Due to this we also get:

$$\begin{aligned} \text{Tr}(e_{t^{1/2}}(b_i)) &= \|e_{t^{1/2}}(b_i)\|_{2,\text{Tr}}^2 \\ &\geq \|\Phi'_i(e_{t^{1/2}}(b_i))\|_{2,\text{Tr}}^2 \geq (1 - \delta'^{1/8})^2 \|e_{t^{1/2}}(b_0)\|_{2,\text{Tr}}^2 \\ &= (1 - \delta'^{1/8})^2 \text{Tr}(e_{t^{1/2}}(b_0)) > (1 - 2\delta'^{1/8}) \text{Tr}(e_{t^{1/2}}(b_0)) \end{aligned}$$

for all $t \in D'$ and all $i = 1, \dots, n$. Let then D'_i be the set of all $t \in D'$ for which $\text{Tr}(e_{t^{1/2}}(b_i)) \leq (1 + \delta'^{1/16}) \text{Tr}(e_{t^{1/2}}(b_0))$. It follows that we have:

$$\begin{aligned}
\delta'^{1/16} \int_{D' \setminus D'_i} \text{Tr}(e_{t^{1/2}}(b_0)) dt &\leq \int_{D' \setminus D'_i} (\text{Tr}(e_{t^{1/2}}(b_i)) - \text{Tr}(e_{t^{1/2}}(b_0))) dt \\
&= \int_{D'} \text{Tr}(e_{t^{1/2}}(b_i)) - \text{Tr}(e_{t^{1/2}}(b_0)) dt \\
&\quad + \int_{D'_i} (\text{Tr}(e_{t^{1/2}}(b_0)) - \text{Tr}(e_{t^{1/2}}(b_i))) dt \\
&\leq \int_{\mathbf{R}_+^* \setminus D'} (\text{Tr}(e_{t^{1/2}}(b_0)) - \text{Tr}(e_{t^{1/2}}(b_i))) dt \\
&\quad + 2\delta'^{1/8} \int_{D'_i} \text{Tr}(e_{t^{1/2}}(b_0)) dt \\
&\leq \int_{\mathbf{R}_+^* \setminus D'} \text{Tr}(e_{t^{1/2}}(b_0)) dt + 2\delta'^{1/8} \\
&\leq 5n\delta'^{1/4} + 2\delta'^{1/8} < 3\delta'^{1/8},
\end{aligned}$$

in which we used, in the previous estimates, the identity

$$\int_{D'} (\text{Tr}(e_{t^{1/2}}(b_i)) - \text{Tr}(e_{t^{1/2}}(b_0))) dt = \int_{\mathbf{R}_+^* \setminus D'} (\text{Tr}(e_{t^{1/2}}(b_0)) - \text{Tr}(e_{t^{1/2}}(b_i))) dt$$

(which follows from the equalities

$$\|b_i\|_{2,\text{Tr}}^2 = \int_{\mathbf{R}_+^*} \text{Tr}(e_{t^{1/2}}(b_i)) dt = \int_{\mathbf{R}_+^*} \text{Tr}(e_{t^{1/2}}(b_0)) dt = \|b_0\|_{2,\text{Tr}}^2$$

and the fact that $5n\delta'^{1/8} < 1$.

It thus follows that if we put $D'' = \bigcap_{i=1}^n D'_i \cap D$ and take into account that $\delta < (5n)^{-32}$, then we get:

$$\begin{aligned}
\int_{D''} \text{Tr}(e_{t^{1/2}}(b_0)) dt &\geq \int_{D \cap D'} \text{Tr}(e_{t^{1/2}}(b_0)) dt - \sum_{i=1}^n \int_{D' \setminus D'_i} \text{Tr}(e_{t^{1/2}}(b_0)) dt \\
&\geq 1 - 5n\delta^{1/4} - 5n\delta'^{1/4} - 3n\delta'^{1/16} \\
&> 1 - 5n\delta^{1/32} > 0.
\end{aligned}$$

Thus $D'' \neq \emptyset$. But if $s > 0$ is such that $s^2 \in D''$ then from the above we have

$$|\text{Tr}(e_s(b_0)) - \text{Tr}(e_s(b_i))| < \delta^{1/16} \text{Tr}(e_s(b_0)),$$

which ends the proof of (ii).

Proof of (iii) in the Theorem. Finally, (iii) follows now immediately from the last inequality above, since we have :

$$\begin{aligned}
\|\Phi_i(e_s(b_i))\|_{2,\text{Tr}} &\geq \|\Phi_0^*(\Phi_i(e_s(b_i)))\|_{2,\text{Tr}} \\
&\geq \|e_s(b_0)\|_{2,\text{Tr}} - \|e_s(b_0) - \Phi'_i(e_s(b_i))\|_{2,\text{Tr}} \\
&> (1 - \delta'^{1/8})\|e_{t^{1/2}}(b_0)\|_{2,\text{Tr}} \\
&> (1 - \delta'^{1/8})(1 + \delta^{1/16})^{-1/2}\|e_{t^{1/2}}(b_i)\|_{2,\text{Tr}} \\
&\geq (1 - \delta^{1/32})\|e_{t^{1/2}}(b_i)\|_{2,\text{Tr}}.
\end{aligned}$$

This ends the proof of the last part of the theorem. \square

Proof of Corollary 0.2. As for the Corollary in the Introduction, it follows readily from the Theorem, by taking $n = 1$, $\Phi_0 = \Phi_1 = \Phi$, once we observe that, since Φ is positive, it is selfadjoint, so

$$\begin{aligned}
\sup\{\|\Phi(x)\|_{2,\text{Tr}} \mid x \in P_1, \|x\|_{2,\text{Tr}} = 1\} \\
= \sup\{\|\Phi(x)\|_{2,\text{Tr}} \mid x \in P_1, x = x^*, \|x\|_{2,\text{Tr}} = 1\},
\end{aligned}$$

and also by noticing that if $x \in P_1$ is such that $x = x^*$ then $\|\Phi(|x|)\|_{2,\text{Tr}} \geq \|\Phi(x)\|_{2,\text{Tr}}$. Indeed, this is because by approximating x by step functions (through spectral calculus) we may assume $x = \sum_i c_i p_i$ for some real scalars c_i and finitely many, mutually orthogonal projections of finite trace p_i . Then, taking into account that $\Phi(p_i), \Phi(p_j) \geq 0$ implies $\text{Tr}(\Phi(p_i)\Phi(p_j)) \geq 0$, we get :

$$\begin{aligned}
\|\Phi(x)\|_{2,\text{Tr}}^2 &= \sum_{i,j} c_i \bar{c}_j \text{Tr}(\Phi(p_i)\Phi(p_j)) \\
&\leq \sum_{i,j} |c_i||\bar{c}_j| \text{Tr}(\Phi(p_i)\Phi(p_j)) = \|\Phi(|x|)\|_{2,\text{Tr}}^2.
\end{aligned}$$

\square

2. APPLICATIONS

We shall apply Theorem 0.1 to a case when the semifinite algebras are in fact commutative. We mention that the noncommutativity will be implicitly present though, through the consideration of the positive maps. Note also that in the proof of the Corollary below, only part (i) in the conclusion of the Theorem is being used. In turn, the proof of this part of the Theorem is relatively short.