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Similarly we have

$$\tilde{\text{wt}} \left( \begin{array}{c} a \\ b \\ + \\ a \\ b \end{array} \right) = -1 \quad \text{and} \quad \tilde{\text{wt}} \left( \begin{array}{c} a \\ a \\ + \\ a \\ a \end{array} \right) = q^{1/2}.$$

We have a similar formula for a negative crossing.

Therefore we see that our graph invariant gives an  $R$ -matrix of the form

$$R_{ij}^{kl} = \begin{cases} q^{1/2} - q^{-1/2} & \text{if } i = k > j = l, \\ -1 & \text{if } i = l \neq j = k, \\ q^{1/2} & \text{if } i = j = k = l, \\ 0 & \text{otherwise;} \end{cases}$$

and

$$(R^{-1})_{ij}^{kl} = \begin{cases} -q^{1/2} + q^{-1/2} & \text{if } i = k < j = l, \\ -1 & \text{if } i = l \neq j = k, \\ q^{-1/2} & \text{if } i = j = k = l, \\ 0 & \text{otherwise;} \end{cases}$$

which coincides with  $-R(q^{1/2})^{-1}$ , where  $R(q^{1/2})$  is the  $R$ -matrix given in [16], replacing  $q$  with  $q^{1/2}$ .

## §5. INVARIANTS CORRESPONDING TO ANTI-SYMETRIC TENSORS

In this section, we will show briefly that our graph invariant gives the quantum link invariant each of its component equipped with an anti-symmetric tensor of the standard  $n$ -dimensional representation of  $SU(n)$ .

Let  $D$  be a link diagram each of its component colored with an integer  $i$  ( $1 \leq i \leq n$ ). This corresponds to the  $i$ -fold anti-symmetric tensor of the standard representation of  $SU(n)$ .

Then  $\langle D \rangle_n$  is defined by

$$\left\langle \begin{array}{c} i \\ \diagup \quad \diagdown \\ j \end{array} \right\rangle_n = \sum_{k=0}^i (-1)^{k+(j+1)i} q^{(i-k)/2} \left\langle \begin{array}{c} i & j+k-i & j \\ & \nearrow & \uparrow \\ j+k & k & i-k \\ & \searrow & \downarrow \\ j & i & \end{array} \right\rangle_n, \quad \text{for } i \leq j$$

and

$$\left\langle \begin{array}{c} i \\ \diagup \quad \diagdown \\ j \end{array} \right\rangle_n = \sum_{k=0}^j (-1)^{k+(i+1)j} q^{(j-k)/2} \left\langle \begin{array}{c} i & i+k-j & j \\ & \nearrow & \uparrow \\ j-k & k & i+k \\ & \searrow & \downarrow \\ j & i & \end{array} \right\rangle_n, \quad \text{for } i > j$$

For a negative crossing, replace  $q$  with  $q^{-1}$ .

Now we will show

**THEOREM 5.1.** *The quantity  $\langle D \rangle_n$  with  $D$  a colored link diagram is invariant under the Reidemeister moves II and III. Thus it is a colored framed link invariant.*

To prove the theorem above, we prepare some lemmas:

**LEMMA 5.2.**

$$\left\langle \begin{array}{c} 1 & i+1 & i \\ & \diagup & \diagdown \\ i & & 1 \\ & \diagdown & \diagup \\ 1 & i+1 & i \\ & \diagup & \diagdown \\ 1 & & 1 \end{array} \right\rangle_n = [n-i-1] \left\langle \begin{array}{c} 1 & i-1 & i \\ & \diagup & \diagdown \\ 1 & & i \\ & \diagdown & \diagup \\ 1 & i-1 & i \\ & \diagup & \diagdown \\ 1 & & 1 \end{array} \right\rangle_n + \left\langle \begin{array}{c} 1 \\ \uparrow \\ i \end{array} \right\rangle_n.$$

*Proof.* The proof of this lemma is similar to that of Lemma 2.4 and we leave it to the reader.  $\square$

LEMMA 5.3.

$$(5.1) \quad \left\{ \begin{array}{c} i \uparrow \\ \diagup \\ j \uparrow \\ \diagdown \\ k \uparrow \\ \diagup \\ i+j \end{array} \right\}_n = \left\{ \begin{array}{c} i \uparrow \\ \diagup \\ j \uparrow \\ \diagdown \\ k \uparrow \\ \diagup \\ i+j \end{array} \right\}_n.$$

*Proof.* It suffices to prove the case  $i = 1$  or  $j = 1$  since we have

$$\begin{aligned} & \left\{ \begin{array}{c} i \uparrow \\ \diagup \\ j \uparrow \\ \diagdown \\ k \uparrow \\ \diagup \\ i+j \end{array} \right\}_n = \frac{1}{[j]} \left\{ \begin{array}{c} i \uparrow \\ \diagup \\ j-1 \uparrow \\ \diagdown \\ 1 \uparrow \\ \diagup \\ j \uparrow \\ \diagdown \\ k \uparrow \\ \diagup \\ i+j \end{array} \right\}_n \\ &= \frac{1}{[j]} \left\{ \begin{array}{c} i \uparrow \\ \diagup \\ j-1 \uparrow \\ \diagdown \\ 1 \uparrow \\ \diagup \\ i+j-1 \end{array} \right\}_n = \frac{1}{[j]} \left\{ \begin{array}{c} i \uparrow \\ \diagup \\ j-1 \uparrow \\ \diagdown \\ 1 \uparrow \\ \diagup \\ i+j-1 \end{array} \right\}_n \end{aligned}$$

and the conclusion follows from the case  $i = 1$  or  $j = 1$  and induction. Here we use Lemma A.1 in the first equality.

We only prove the case  $j = 1$  and  $i < k$  since the remaining case is similar. From the definition, the left hand side of (5.1) with  $j = 1$  equals

$$\sum_{l=0}^{i+1} (-1)^{l+(k+1)(i+1)} q^{(i-l+1)/2} \left\{ \begin{array}{c} i \uparrow \\ \diagup \\ 1 \uparrow \\ \diagdown \\ k+l-i-1 \uparrow \\ \diagup \\ k+l \uparrow \\ \diagdown \\ l \uparrow \\ \diagup \\ i+1 \end{array} \right\}_n.$$

The right hand side becomes

$$\begin{aligned}
 & \sum_{l=0}^i (-1)^{l+(k+1)(i+1)} q^{(i-l+1)/2} \left\{ \begin{array}{c} i \\ k+l \\ k \\ \end{array} \right. \left. \begin{array}{c} 1 \\ k-1 \\ k \\ l \\ k+l-i \\ i-l \\ i \\ i+1 \\ n \end{array} \right\} \\
 & + \sum_{l=0}^i (-1)^{l+(k+1)i+k} q^{(i-l)/2} \left\{ \begin{array}{c} i \\ k+l \\ k \\ \end{array} \right. \left. \begin{array}{c} 1 \\ k+1 \\ k \\ l \\ k+l-i \\ i-l \\ i \\ i+1 \\ n \end{array} \right\}.
 \end{aligned}$$

Sliding the bar colored with  $l$  using Lemma 2.6, the first diagram becomes

$$\begin{aligned}
 & \left\{ \begin{array}{c} i \\ k+l \\ k \\ \end{array} \right. \left. \begin{array}{c} 1 \\ k-1 \\ k \\ l \\ k+l-i \\ i-l \\ i-l+1 \\ i+1 \\ n \end{array} \right\} = [i-l] \left\{ \begin{array}{c} i \\ k+l \\ k \\ \end{array} \right. \left. \begin{array}{c} 1 \\ k+1 \\ k \\ l \\ k+l-i \\ i-l+1 \\ i+1 \\ n \end{array} \right\} + \left\{ \begin{array}{c} i \\ k+l \\ k \\ \end{array} \right. \left. \begin{array}{c} 1 \\ k+l-i-1 \\ k \\ l \\ k+l-i \\ i-l+1 \\ i+1 \\ n \end{array} \right\} \\
 & = [i-l] \left\{ \begin{array}{c} i \\ k+l \\ k \\ \end{array} \right. \left. \begin{array}{c} 1 \\ k+1 \\ k \\ l \\ k+l-i \\ i-l+1 \\ i+1 \\ n \end{array} \right\} + \left\{ \begin{array}{c} i \\ k+l \\ k \\ \end{array} \right. \left. \begin{array}{c} 1 \\ k+l-i-1 \\ k \\ l \\ k+l-i \\ i-l+1 \\ i+1 \\ n \end{array} \right\},
 \end{aligned}$$

where the first equality follows from Lemma A.7 below.

The second diagram turns out to be

$$\left\{ \begin{array}{c} i \\ \uparrow \\ 1 \\ \uparrow \\ k \\ \uparrow \\ k+l-i \\ \curvearrowright \\ k+l \\ \uparrow \\ k \\ \uparrow \\ l \\ \uparrow \\ i-l \\ \curvearrowright \\ i-l+1 \\ \uparrow \\ i+1 \end{array} \right\}_n = [i-l+1] \left\{ \begin{array}{c} i \\ \uparrow \\ 1 \\ \uparrow \\ k \\ \uparrow \\ k+l-i \\ \curvearrowright \\ k+l \\ \uparrow \\ k \\ \uparrow \\ l \\ \uparrow \\ i+1 \\ \uparrow \\ i-l+1 \end{array} \right\}_n.$$

Therefore the right hand side of (5.1) becomes

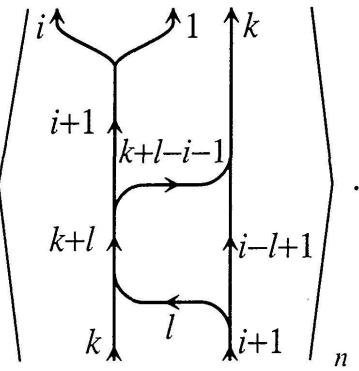
$$\left\{ \sum_{l=0}^i (-1)^{l+(k+1)(i+1)} q^{(i-l+1)/2} [i-l] + \sum_{l=0}^i (-1)^{l+(k+1)i+k} q^{(i-l)/2} [i-l+1] \right\}$$

$$\times \left\{ \begin{array}{c} i \\ \uparrow \\ 1 \\ \uparrow \\ k \\ \uparrow \\ k+l-i \\ \curvearrowright \\ k+l \\ \uparrow \\ k \\ \uparrow \\ l \\ \uparrow \\ i+1 \\ \uparrow \\ i-l+1 \end{array} \right\}_n$$

$$+ \sum_{l=0}^i (-1)^{l+(k+1)(i+1)} q^{(i-l+1)/2} \left\{ \begin{array}{c} i \\ \uparrow \\ 1 \\ \uparrow \\ k \\ \uparrow \\ i+1 \\ \uparrow \\ k+l-i-1 \\ \curvearrowright \\ k+l \\ \uparrow \\ k \\ \uparrow \\ l \\ \uparrow \\ i+1 \\ \uparrow \\ i-l+1 \end{array} \right\}_n$$

$$= \sum_{l=0}^i (-1)^{l+(k+1)(i+1)+1} \left\{ \begin{array}{c} i \\ \uparrow \\ 1 \\ \uparrow \\ k \\ \uparrow \\ k+l-i \\ \curvearrowright \\ k+l \\ \uparrow \\ k \\ \uparrow \\ l \\ \uparrow \\ i+1 \\ \uparrow \\ i-l+1 \end{array} \right\}_n$$

$$+ \sum_{l=0}^i (-1)^{l+(k+1)(i+1)} q^{(i-l+1)/2}$$



We finally see that the right hand side of (5.1) minus the left hand side equals

$$\begin{aligned} & \sum_{l=0}^i (-1)^{l+(k+1)(i+1)+1} \left\langle \begin{array}{c} i \\ \uparrow \\ k+l \\ \uparrow \\ k \\ \uparrow \\ k+l-i \\ \uparrow \\ i-l+1 \\ \uparrow \\ i+1 \end{array} \right\rangle_n - (-1)^{i+1+(k+1)(i+1)} \left\langle \begin{array}{c} i \\ \uparrow \\ i+1 \\ \uparrow \\ k+i+1 \\ \uparrow \\ i+1 \end{array} \right\rangle_n \\ & = \sum_{l=0}^{i+1} (-1)^{l+(k+1)(i+1)+1} \left\langle \begin{array}{c} i \\ \uparrow \\ k+l \\ \uparrow \\ k \\ \uparrow \\ k+l-i \\ \uparrow \\ i-l+1 \\ \uparrow \\ i+1 \end{array} \right\rangle_n = 0. \end{aligned}$$

Here the last equality follows from Lemma A.9, completing the proof.  $\square$

### PROOF OF THEOREM 5.1

*The invariance under the Reidemeister move II.* We will first show

$$\left\langle \begin{array}{c} i \\ \uparrow \\ j \end{array} \right\rangle_n = \left\langle \begin{array}{c} i \\ \uparrow \\ j \end{array} \right\rangle_n.$$

It suffices to show the case  $i = 1$  from Lemmas A.1 and 5.3 since

$$\left\langle \begin{array}{c} i \\ j \end{array} \right\rangle_n = \frac{1}{[i]} \left\langle \begin{array}{c} i \\ j \\ 1 \\ i-1 \end{array} \right\rangle_n = \frac{1}{[i]} \left\langle \begin{array}{c} i \\ j \\ 1 \\ i-1 \\ 1 \\ i-1 \end{array} \right\rangle_n$$

$$= \frac{1}{[i]} \left\langle \begin{array}{c} i \\ j \\ 1 \\ i-1 \\ 1 \\ i-1 \end{array} \right\rangle_n = \frac{1}{[i]} \left\langle \begin{array}{c} i \\ j \\ 1 \\ i-1 \\ 1 \\ i-1 \end{array} \right\rangle_n$$

$$= \left\langle \begin{array}{c} i \\ j \end{array} \right\rangle_n,$$

where the second equality follows from Lemma 5.3 and the fourth by induction on  $i$ . Now we have

$$\left\langle \begin{array}{c} 1 \\ j \\ i \end{array} \right\rangle_n = \left\langle \begin{array}{c} 1 \\ j \\ j-1 \\ j \\ 1 \\ j \\ 1 \end{array} \right\rangle_n - q^{1/2} \left\langle \begin{array}{c} 1 \\ j \\ j-1 \\ j \\ 1 \\ j+1 \\ j+1 \end{array} \right\rangle_n - q^{-1/2} \left\langle \begin{array}{c} 1 \\ j \\ j-1 \\ j \\ j+1 \\ j+1 \\ j \end{array} \right\rangle_n + \left\langle \begin{array}{c} 1 \\ j \\ j+1 \\ j \\ j+1 \\ j \\ 1 \end{array} \right\rangle_n$$

$$\begin{aligned}
&= \left\langle \begin{array}{c} 1 \uparrow \\ j \uparrow \\ 1 \end{array} \right\rangle_n + (-q^{1/2}[j] - q^{-1/2}[j] + [j+1]) \left\langle \begin{array}{c} 1 \uparrow \\ j+1 \uparrow \\ 1 \end{array} \right\rangle_n \\
&= \left\langle \begin{array}{c} 1 \uparrow \\ j \uparrow \end{array} \right\rangle_n + ([j-1] - q^{1/2}[j] - q^{-1/2}[j] + [j+1]) \left\langle \begin{array}{c} 1 \uparrow \\ j+1 \uparrow \\ j \end{array} \right\rangle_n \\
&= \left\langle \begin{array}{c} 1 \uparrow \\ j \uparrow \end{array} \right\rangle_n,
\end{aligned}$$

where we use Lemma A.4 in the third equality.

Next we will show

$$\left\langle \begin{array}{c} i \uparrow \\ j \downarrow \end{array} \right\rangle_n = \left\langle \begin{array}{c} i \uparrow \\ j \downarrow \end{array} \right\rangle_n.$$

It also suffices to show the case  $i = 1$  as above. We have

$$\begin{aligned}
&\left\langle \begin{array}{c} 1 \uparrow \\ j \downarrow \end{array} \right\rangle_n \\
&= \left\langle \begin{array}{c} 1 \uparrow \\ j \downarrow \\ j-1 \uparrow \\ j-1 \downarrow \\ 1 \end{array} \right\rangle_n - q^{1/2} \left\langle \begin{array}{c} 1 \uparrow \\ j \downarrow \\ j+1 \uparrow \\ j-1 \downarrow \\ 1 \end{array} \right\rangle_n - q^{-1/2} \left\langle \begin{array}{c} 1 \uparrow \\ j \downarrow \\ j-1 \uparrow \\ j+1 \downarrow \\ 1 \end{array} \right\rangle_n + \left\langle \begin{array}{c} 1 \uparrow \\ j \downarrow \\ j+1 \uparrow \\ j+1 \downarrow \\ 1 \end{array} \right\rangle_n \\
&= ([n-j+1] - q^{1/2}[n-j] - q^{-1/2}[n-j]) \left\langle \begin{array}{c} 1 \uparrow \\ j-1 \downarrow \\ j \end{array} \right\rangle_n + \left\langle \begin{array}{c} 1 \uparrow \\ j \downarrow \\ j+1 \uparrow \\ j+1 \downarrow \\ 1 \end{array} \right\rangle_n
\end{aligned}$$

$$\begin{aligned}
&= \left( [n-j+1] - q^{1/2}[n-j] - q^{-1/2}[n-j] + [n-j-1] \right) \left\langle \begin{array}{c} 1 \uparrow \\ \diagdown j \\ 1 \end{array} \right\rangle_n \\
&\quad + \left\langle \begin{array}{c} 1 \uparrow \\ \diagdown j \\ j \end{array} \right\rangle_n \\
&= \left\langle \begin{array}{c} 1 \uparrow \\ \diagdown j \\ j \end{array} \right\rangle_n,
\end{aligned}$$

where we use Lemma 5.2 in the third equality. Now the proof for the Reidemeister move II is complete.

*The invariance under the Reidemeister move III.* This is proved by repeated application of Lemma 5.3 and details are omitted. See the proof of Theorem 3.1.

#### A. APPENDIX

In this appendix, we give proofs of lemmas used in the previous section.

LEMMA A.1.

$$\left\langle \begin{array}{c} \uparrow i \\ j \curvearrowleft \\ \diagdown i \\ i-j \end{array} \right\rangle_n = [i] \left\langle \begin{array}{c} \uparrow i \\ j \end{array} \right\rangle_n$$

for  $i \geq j \geq 0$ .

*Proof.* The proof for  $j = 1$  is similar to that of Lemma 2.2 and omitted. For  $j > 1$  we have

$$\begin{aligned}
\left\langle \begin{array}{c} \uparrow i \\ j \curvearrowleft \\ \diagdown i \\ i-j \end{array} \right\rangle_n &= \frac{1}{[j]} \left\langle \begin{array}{c} \uparrow i \\ j-1 \curvearrowleft \\ \diagdown i \\ i-j \\ j \curvearrowleft \\ 1 \end{array} \right\rangle_n = \frac{1}{[j]} \left\langle \begin{array}{c} \uparrow i \\ j-1 \curvearrowleft \\ \diagdown i \\ i-j+1 \\ 1 \curvearrowleft \\ i-j \\ i-j+1 \end{array} \right\rangle_n \\
&= \frac{[i-j+1]}{[j]} \left\langle \begin{array}{c} \uparrow i \\ j-1 \curvearrowleft \\ \diagdown i \\ i-j+1 \end{array} \right\rangle_n = \frac{[i-j+1]}{[j]} \begin{bmatrix} i \\ j-1 \end{bmatrix} \left\langle \begin{array}{c} \uparrow i \\ j \end{array} \right\rangle_n = [i] \left\langle \begin{array}{c} \uparrow i \\ j \end{array} \right\rangle_n,
\end{aligned}$$

where the second equality follows from Lemma 2.6 and the fourth by induction. The proof is complete.  $\square$