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Artikel: HOMFLY POLYNOMIAL VIA AN INVARIANT OF COLORED PLANE GRAPHS

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Kapitel: §5. Invariants corresponding to anti-symetric tensors

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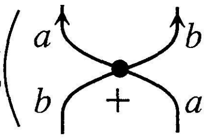
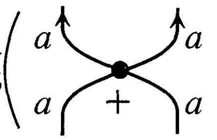
Similarly we have

$\widetilde{\mathrm{wt}}\Big($ 

$\Big) = -1$ and $\widetilde{\mathrm{wt}}\Big($ 

$\Big) = q^{1/2}.$

We have a similar formula for a negative crossing.

Therefore we see that our graph invariant gives an R-matrix of the form

$$R_{ij}^{kl} = \begin{cases} q^{1/2} - q^{-1/2} & \text{if } i = k > j = l, \\ -1 & \text{if } i = l \neq j = k, \\ q^{1/2} & \text{if } i = j = k = l, \\ 0 & \text{otherwise;} \end{cases}$$

and

$$(R^{-1})_{ij}^{kl} = \begin{cases} -q^{1/2} + q^{-1/2} & \text{if } i = k < j = l, \\ -1 & \text{if } i = l \neq j = k, \\ q^{-1/2} & \text{if } i = j = k = l, \\ 0 & \text{otherwise;} \end{cases}$$

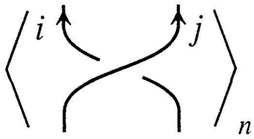
which coincides with $-R(q^{1/2})^{-1}$, where $R(q^{1/2})$ is the R-matrix given in [16], replacing q with $q^{1/2}$.

§5. Invariants corresponding to anti-symetric tensors

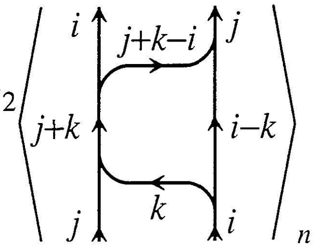
In this section, we will show briefly that our graph invariant gives the quantum link invariant each of its component equipped with an anti-symmetric tensor of the standard n-dimensional representation of $SU(n)$.

Let D be a link diagram each of its component colored with an integer i $(1 \leq i \leq n)$. This corresponds to the i-fold anti-symmetric tensor of the standard representation of $SU(n)$.

Then $\langle D\rangle_n$ is defined by

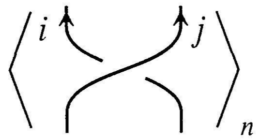


$$= \sum_{k=0}^{i}(-1)^{k+(j+1)i}q^{(i-k)/2}$$

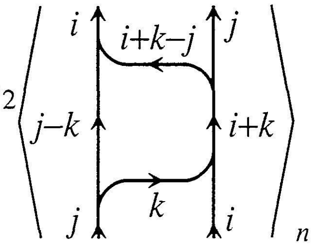


, for $i \leq j$

and



$$= \sum_{k=0}^{j}(-1)^{k+(i+1)j}q^{(j-k)/2}$$



, for $i > j$

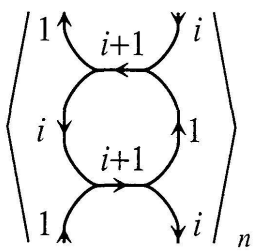
For a negative crossing, replace q with q^{-1}.

Now we will show

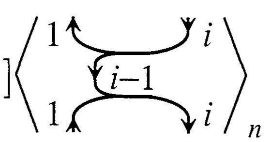
THEOREM 5.1. *The quantity $\langle D\rangle_n$ with D a colored link diagram is invariant under the Reidemeister moves II and III. Thus it is a colored framed link invariant.*

To prove the theorem above, we prepare some lemmas:

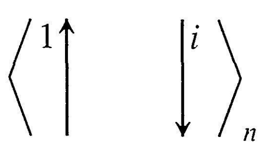
LEMMA 5.2.



$$= [n-i-1]$$



$$+$$



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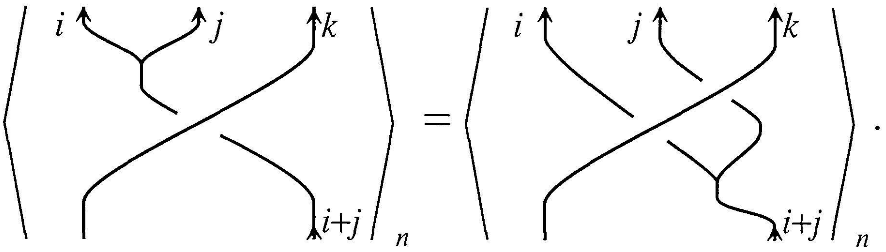
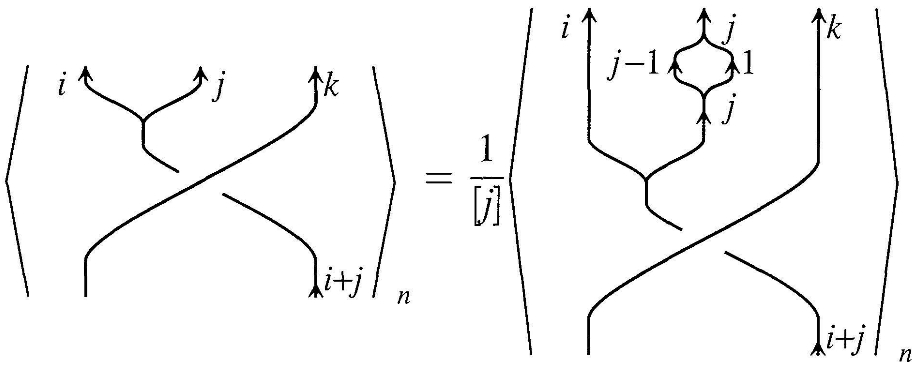
Proof. The proof of this lemma is similar to that of Lemma 2.4 and we leave it to the reader. □

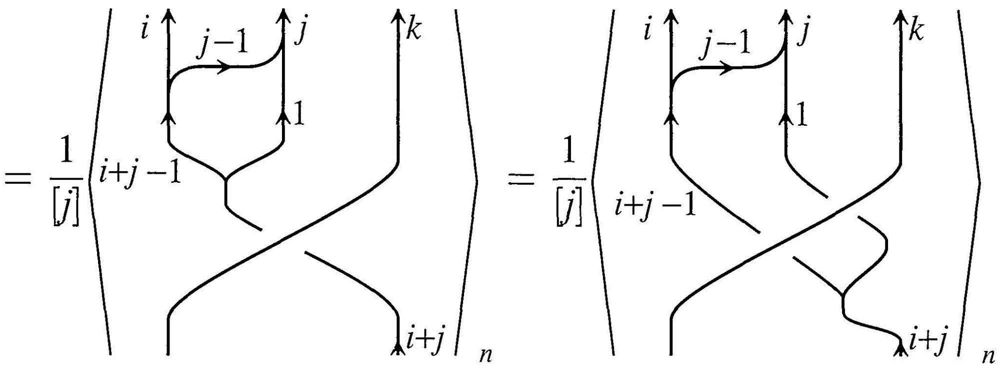
LEMMA 5.3.

(5.1)



Proof. It suffices to prove the case $i = 1$ or $j = 1$ since we have

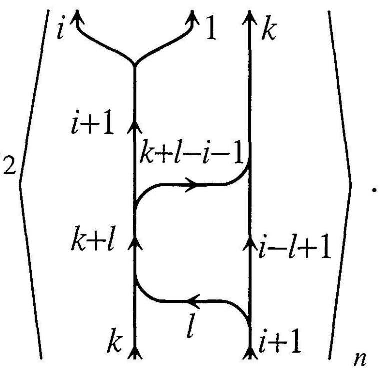




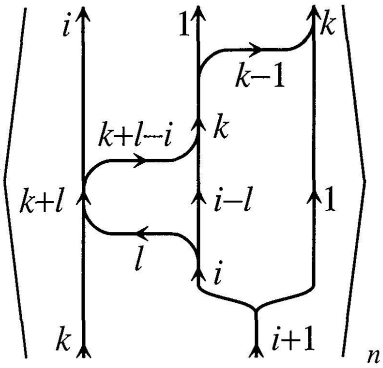
and the conclusion follows from the case $i = 1$ or $j = 1$ and induction. Here we use Lemma A.1 in the first equality.

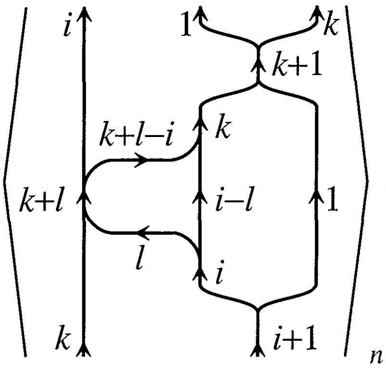
We only prove the case $j = 1$ and $i < k$ since the remaining case is similar. From the definition, the left hand side of (5.1) with $j = 1$ equals

$$\sum_{l=0}^{i+1}(-1)^{l+(k+1)(i+1)}q^{(i-l+1)/2}$$



The right hand side becomes

$$\sum_{l=0}^{i}(-1)^{l+(k+1)(i+1)}q^{(i-l+1)/2}$$


$$+\sum_{l=0}^{i}(-1)^{l+(k+1)i+k}q^{(i-l)/2}$$


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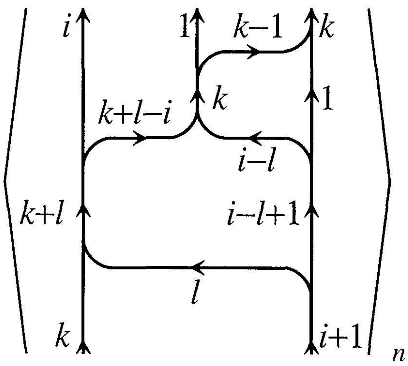
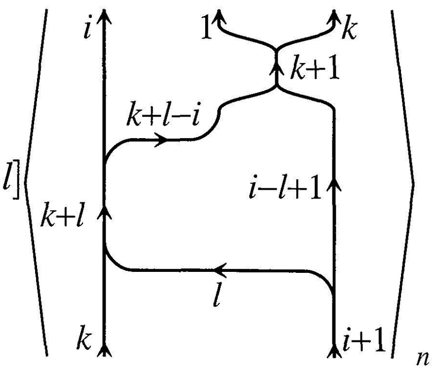
Sliding the bar colored with l using Lemma 2.6, the first diagram becomes



$= [i-l]$


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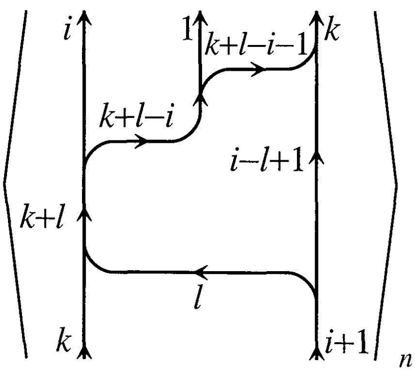
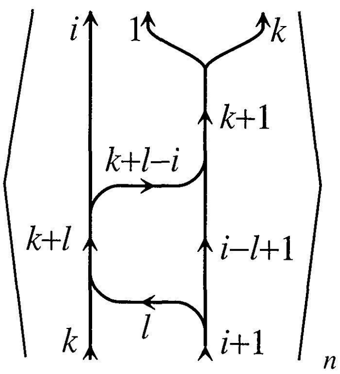
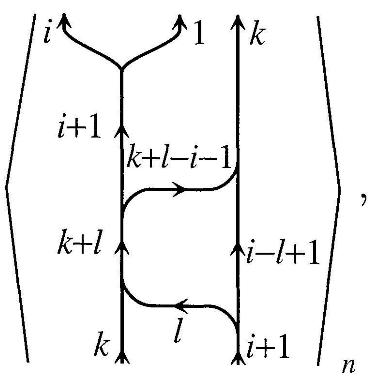


$= [i-l]$


$+$


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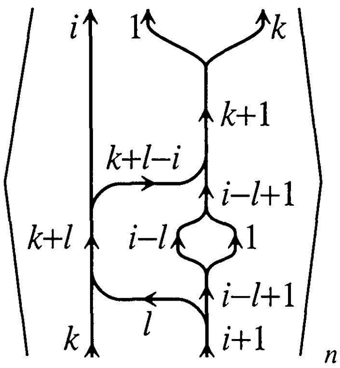
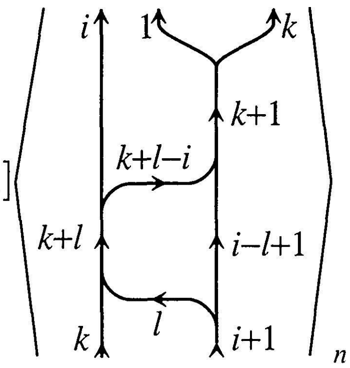
where the first equality follows from Lemma A.7 below.

The second diagram turns out to be



$= [i - l + 1]$

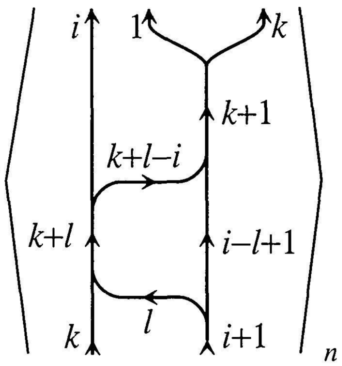


.

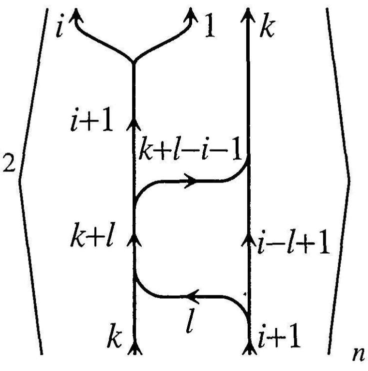
Therefore the right hand side of (5.1) becomes

$$\left\{\sum_{l=0}^{i}(-1)^{l+(k+1)(i+1)}q^{(i-l+1)/2}[i-l] + \sum_{l=0}^{i}(-1)^{l+(k+1)i+k}q^{(i-l)/2}[i-l+1]\right\}$$

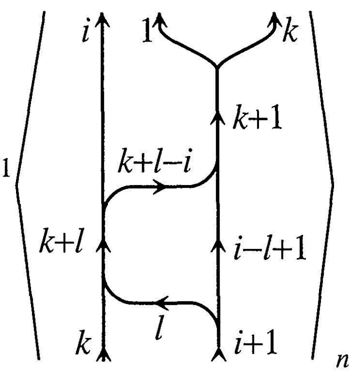
$\times$

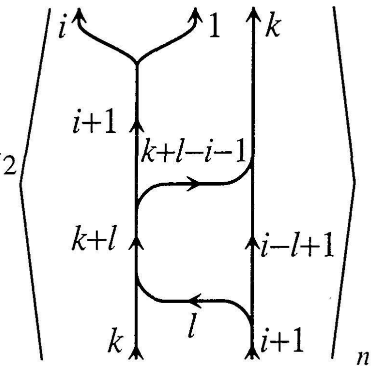


$$+\sum_{l=0}^{i}(-1)^{l+(k+1)(i+1)}q^{(i-l+1)/2}$$



$$= \sum_{l=0}^{i}(-1)^{l+(k+1)(i+1)+1}$$



$$+\sum_{l=0}^{i}(-1)^{l+(k+1)(i+1)}q^{(i-l+1)/2}$$ 

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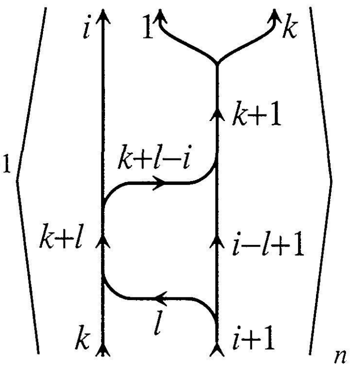
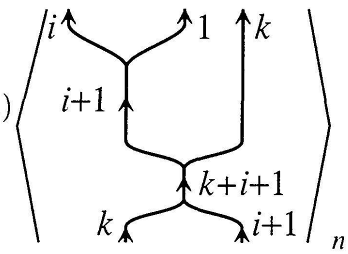
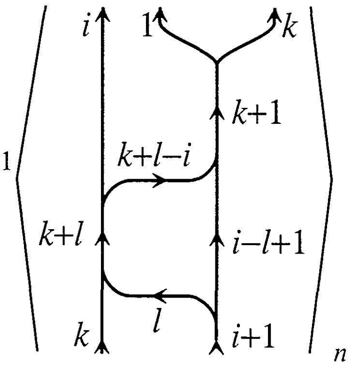
We finally see that the right hand side of (5.1) minus the left hand side equals

$$\sum_{l=0}^{i}(-1)^{l+(k+1)(i+1)+1}$$ 

$$-\,(-1)^{i+1+(k+1)(i+1)}$$ 

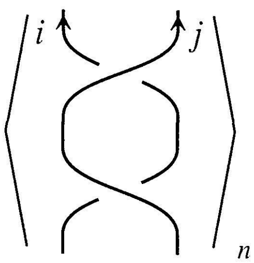
$$=\sum_{l=0}^{i+1}(-1)^{l+(k+1)(i+1)+1}$$ 

$= 0\,.$

Here the last equality follows from Lemma A.9, completing the proof. □

PROOF OF THEOREM 5.1

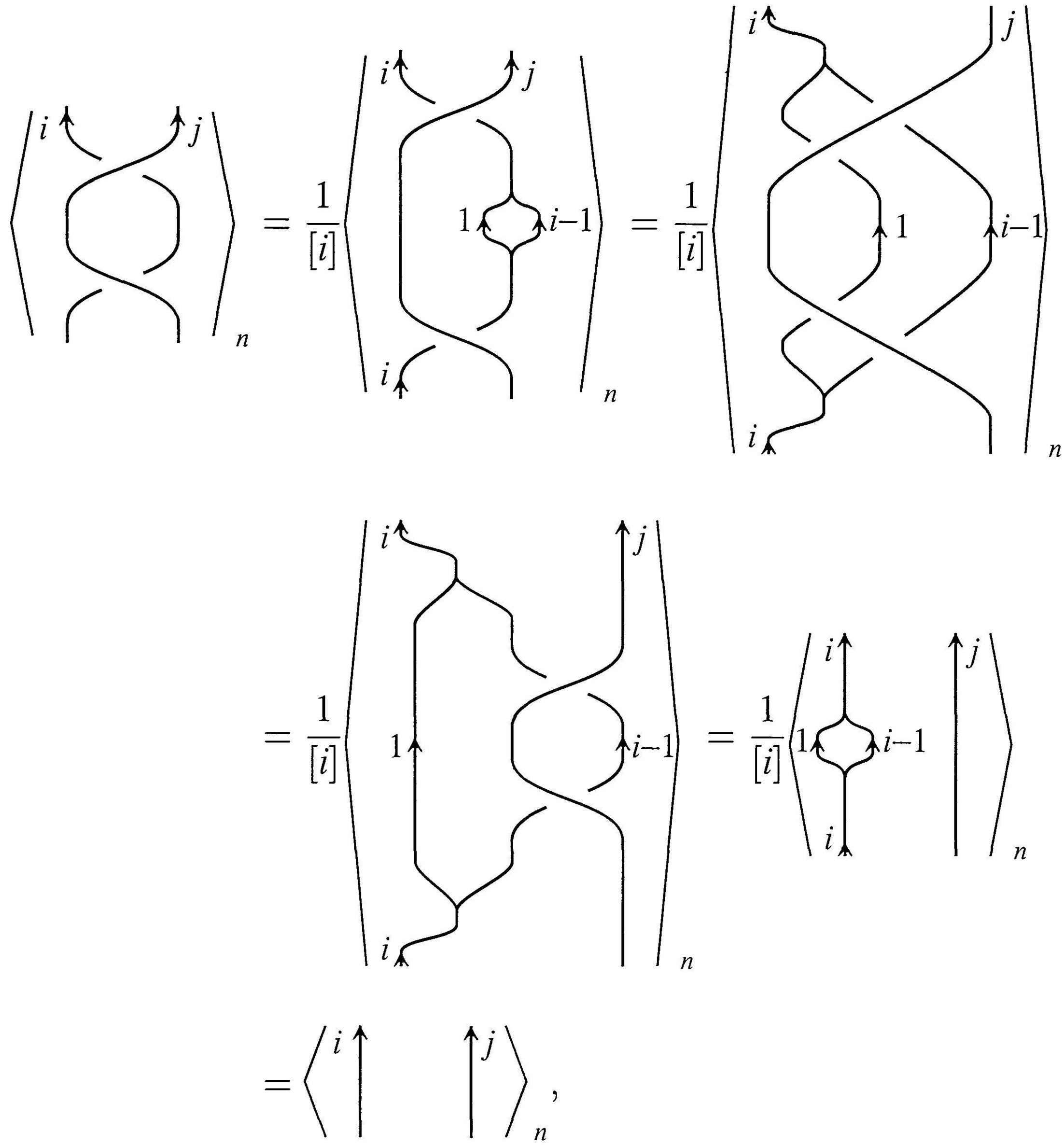
The invariance under the Reidemeister move II. We will first show



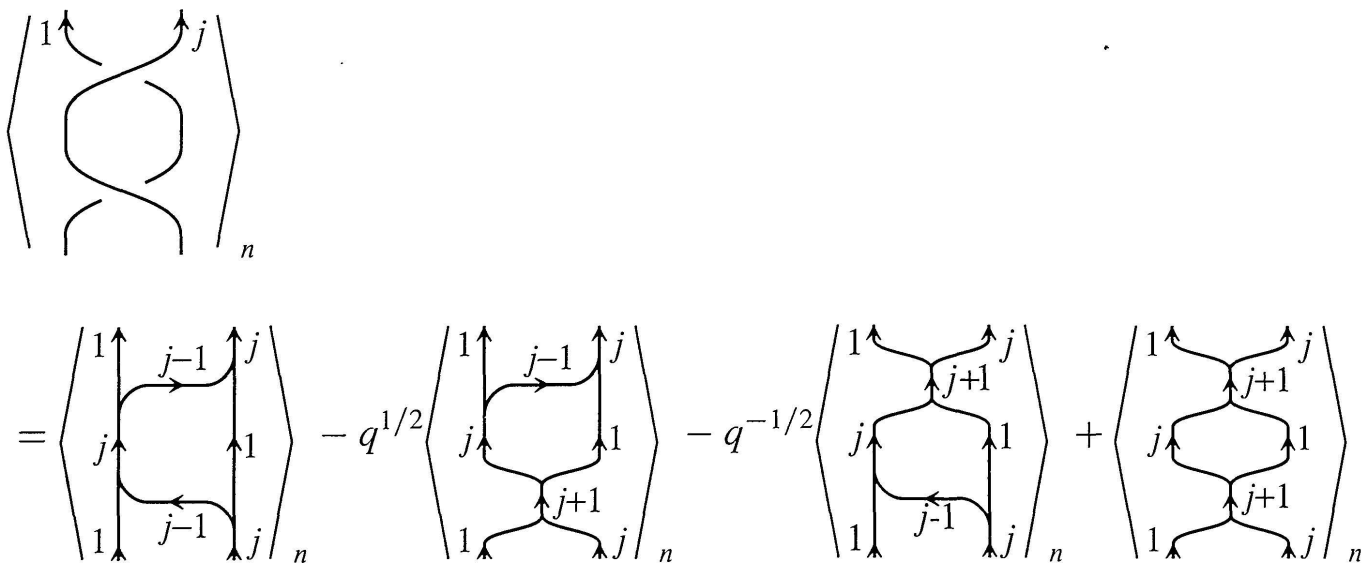
$=$ 

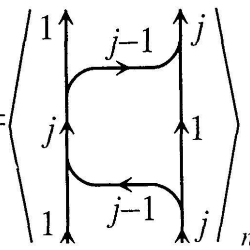
.

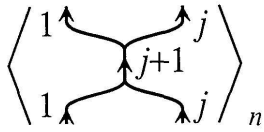
It suffices to show the case $i = 1$ from Lemmas A.1 and 5.3 since

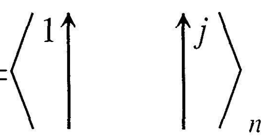


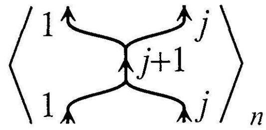
where the second equality follows from Lemma 5.3 and the fourth by induction on i. Now we have

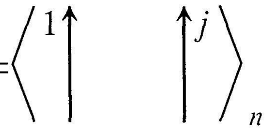


$=$ 

$+\left(-q^{1/2}[j]-q^{-1/2}[j]+[j+1]\right)$ 

$=$ 

$+\left([j-1]-q^{1/2}[j]-q^{-1/2}[j]+[j+1]\right)$ 

$=$ 

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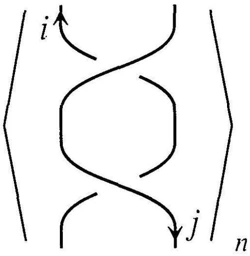
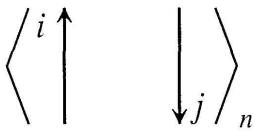
where we use Lemma A.4 in the third equality.

Next we will show



$=$ 

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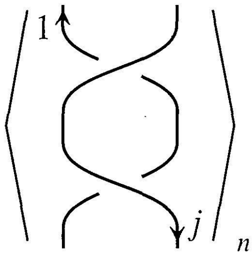
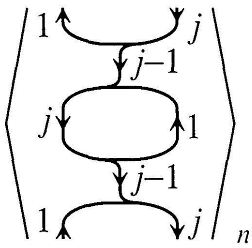
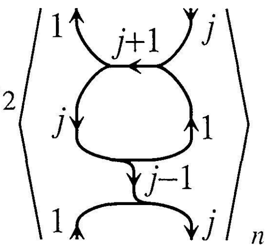
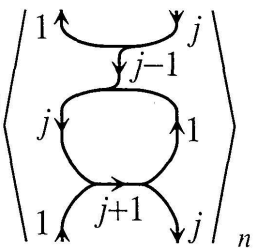
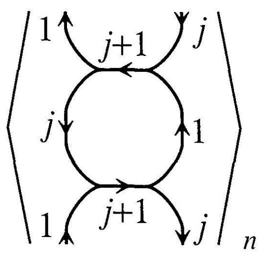
It also suffices to show the case $i=1$ as above. We have

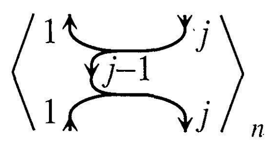


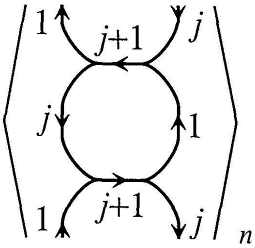
$=$ 

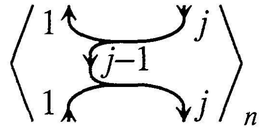
$-q^{1/2}$ 

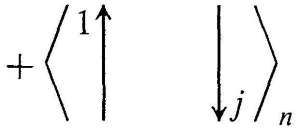
$-q^{-1/2}$ 

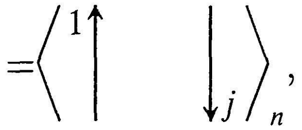
$+$ 

$=\left([n-j+1]-q^{1/2}[n-j]-q^{-1/2}[n-j]\right)$ 

$+$ 

$$= \left([n-j+1] - q^{1/2}[n-j] - q^{-1/2}[n-j] + [n-j-1]\right)$$ 





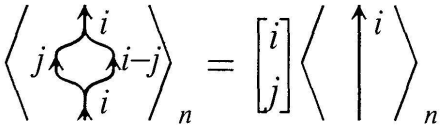
where we use Lemma 5.2 in the third equality. Now the proof for the Reidemeister move II is complete.

The invariance under the Reidemeister move III. This is proved by repeated application of Lemma 5.3 and details are omitted. See the proof of Theorem 3.1.

A. APPENDIX

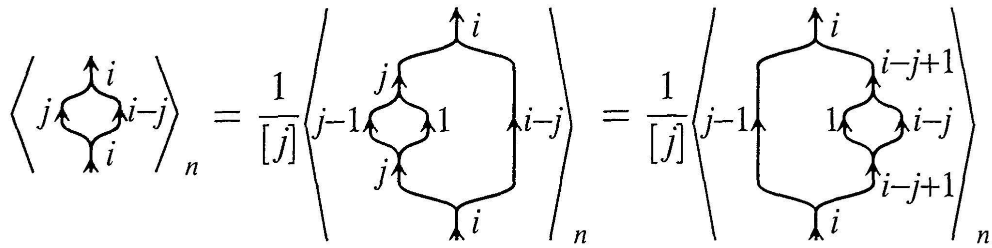
In this appendix, we give proofs of lemmas used in the previous section.

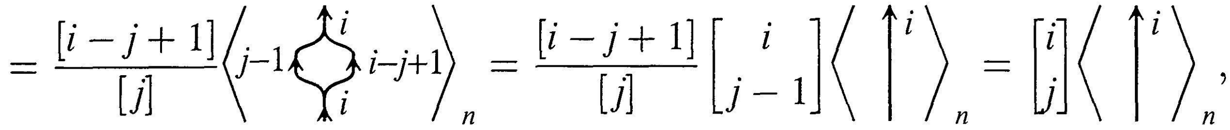
LEMMA A.1.



for $i \geq j \geq 0$.

Proof. The proof for $j = 1$ is similar to that of Lemma 2.2 and omitted. For $j > 1$ we have





where the second equality follows from Lemma 2.6 and the fourth by induction. The proof is complete. □