

Zeitschrift:	L'Enseignement Mathématique
Herausgeber:	Commission Internationale de l'Enseignement Mathématique
Band:	44 (1998)
Heft:	3-4: L'ENSEIGNEMENT MATHÉMATIQUE
 Artikel:	RECENT DEVELOPMENTS ON SERRE'S MULTIPLICITY CONJECTURES: GABBER'S PROOF OF THE NONNEGATIVITY CONJECTURE
Autor:	ROBERTS, Paul C.
Kapitel:	4. SOME EXAMPLES
DOI:	https://doi.org/10.5169/seals-63907

Nutzungsbedingungen

Die ETH-Bibliothek ist die Anbieterin der digitalisierten Zeitschriften auf E-Periodica. Sie besitzt keine Urheberrechte an den Zeitschriften und ist nicht verantwortlich für deren Inhalte. Die Rechte liegen in der Regel bei den Herausgebern beziehungsweise den externen Rechteinhabern. Das Veröffentlichen von Bildern in Print- und Online-Publikationen sowie auf Social Media-Kanälen oder Webseiten ist nur mit vorheriger Genehmigung der Rechteinhaber erlaubt. [Mehr erfahren](#)

Conditions d'utilisation

L'ETH Library est le fournisseur des revues numérisées. Elle ne détient aucun droit d'auteur sur les revues et n'est pas responsable de leur contenu. En règle générale, les droits sont détenus par les éditeurs ou les détenteurs de droits externes. La reproduction d'images dans des publications imprimées ou en ligne ainsi que sur des canaux de médias sociaux ou des sites web n'est autorisée qu'avec l'accord préalable des détenteurs des droits. [En savoir plus](#)

Terms of use

The ETH Library is the provider of the digitised journals. It does not own any copyrights to the journals and is not responsible for their content. The rights usually lie with the publishers or the external rights holders. Publishing images in print and online publications, as well as on social media channels or websites, is only permitted with the prior consent of the rights holders. [Find out more](#)

Download PDF: 24.01.2026

ETH-Bibliothek Zürich, E-Periodica, <https://www.e-periodica.ch>

4. SOME EXAMPLES

We give here three simple examples of the type of bigraded ring which might result from this construction. Each of these examples is obtained by taking a birational map from a regular scheme to $\text{Spec}(R/\mathfrak{q})$, and the last two are simple resolutions of singularities.

We first summarize the construction up to this point. We began with a regular local ring R and prime ideals \mathfrak{p} and \mathfrak{q} . We then took a regular subscheme Z' of $\text{Proj}(R[X_0, \dots, X_n])$ which was generically finite over $\text{Spec}(R/\mathfrak{q})$. The next step was to replace $R[X_0, \dots, X_n]$ with the associated graded ring of I tensored with $R/\mathfrak{m} = k$. The sheaf $\mathcal{O}_{Y'}$ defined by $B = R/\mathfrak{p}[X_0, \dots, X_n]$ was then replaced with the sheaf \mathcal{M} defined by $gr_I(B)$, again tensored with k . The assumption of regularity implies that I/I^2 is locally free over A/I ; denote its rank r . Then the dimension of \mathcal{M} is at most r , and it is equal to r if and only if we had $\dim(R/\mathfrak{p}) + \dim(R/\mathfrak{q}) = \dim(R)$. We note that the fiber Z'_s of Z' over the maximal ideal of R has dimension at most $\dim(R/\mathfrak{q}) - 1$, but apart from that we do not know much about it. It is the projective scheme defined by the graded ring $(A/I) \otimes_R k$, which is the part of degree zero in the grading in the second component of the bigraded ring we are considering.

For the first example, let R have dimension four, let t, u, v, w be a regular system of parameters, and define the prime ideals \mathfrak{p} and \mathfrak{q} by letting $\mathfrak{p} = (t, u)$ and $\mathfrak{q} = (v, w)$. In this case, $\text{Spec}(R/\mathfrak{q})$ is already regular, and we can simply take the projective scheme $\text{Proj}(R[X]) = \text{Spec}(R)$.

For a slightly more complicated example, consider the subscheme of the projective space $\text{Proj}(R[X, Y])$ defined by the ideal I generated by v, w , and $Xu - Yt$. Then Z' is the blow up of R/\mathfrak{q} at the point defined by the maximal ideal, and the fiber over s is projective space of dimension 1. One could define similar examples in higher dimension.

For a third example, let R have dimension 2, and let I be generated by $Xu - Yt, Zu - Xt, X^2 - YZ$. The projective space P has dimension 2, and the fiber over the maximal ideal has codimension one in $\text{Proj}(k[X, Y, Z])$ and thus has dimension one. The sheaf defined by I/I^2 has rank 2, but I is minimally generated by three elements.

In the above examples it was not really necessary to reduce to projective space since the original quotients R/\mathfrak{q} were regular. We next consider an example where the original scheme is not regular. Let \mathfrak{m} be minimally generated by t, u , and let \mathfrak{q} be the principal ideal generated by $t^2 - u^3$.

We can resolve the singularity by letting Z' be defined by the ideal in $R[X, Y]$ generated by $t^2 - u^3$, $uX - tY$, $X^2 - uY^2$. The fiber Z'_s in this case is $\text{Proj}(k[X, Y]/(X^2))$.

Finally, we consider the case where \mathfrak{q} is the determinantal ideal in R of dimension 4 generated by $wu - t^2$, $wv - tu$, and $tv - u^2$. In this case the resolution can be found by taking the ideal I in $R[X, Y, Z, W]$ generated by the following elements :

$$Z^2 - YW, YZ - XW, Y^2 - XZ, uW - vZ, uZ - vY, uY - vX, u^2 - tv, \\ tW - vY, tZ - vX, tY - uX, tu - wv, t^2 - wu, wW - vX, wZ - uX, wY - tX.$$

The fiber over the maximal ideal is a determinantal subvariety of dimension 1.

In a later section we will return to these examples and consider the question of computing the Euler characteristics $\chi_{E_s}(\mathcal{O}_{Z'_s}, \mathcal{M})$ for sheaves \mathcal{M} defined as above by certain prime ideals \mathfrak{p} of R .

5. HILBERT POLYNOMIALS OF BIGRADED MODULES

In section 2 we showed how the Serre spectral sequence can be used to express the Euler characteristic defined by a Koszul complex in terms of the Samuel multiplicity. In this section we show that similar results hold in the present situation. We now let C denote the bigraded ring which we previously denoted $gr_I(A) \otimes_R k$, where $C_{i,j}$ consists of the elements of $(I^j/I^{j+1}) \otimes k$ of degree i . Thus in our present notation, $E_s = \text{Proj}(C)$, where the grading on C is that in the first coordinate. Let C_0 denote the subring $\bigoplus_i (C_{i,0})$. Let r be the rank of I/I^2 , and let M be a bigraded module defining a sheaf \mathcal{M} on E_s of dimension at most r ; we define the dimension of M to be the dimension of the associated sheaf. We consider the question of computing the Euler characteristic $\chi_{E_s}(\mathcal{O}_{Z'_s}, \mathcal{M})$, which we also denote $\chi(C_0, M)$.

Let

$$0 \rightarrow F_k \rightarrow \cdots \rightarrow F_1 \rightarrow F_0 \rightarrow C_0 \rightarrow 0$$

be a complex of bigraded modules which defines a locally free resolution of C_0 over C . For any finitely generated bigraded module N , we let $P_N(m, n)$ by the Hilbert polynomial of N ; more precisely, we define P_N to be the polynomial in two variables such that

$$P_N(m, n) = \sum_{i=0}^{n-1} \text{length}(N_{m,i})$$