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RECENT DEVELOPMENTS ON SERRE'S MULTIPLICITY CONJECTURES: GABBER'S PROOF OF THE NONNEGATIVITY CONJECTURE

by Paul C. ROBERTS

These notes are based on talks given at the Encuentro de Geometría Algebraica y Álgebra Conmutativa in Guanajuato in August 1997. They describe recent developments in the questions on intersection multiplicities, particularly Gabber's recent proof of Serre's conjecture that intersection multiplicities over regular local rings are non-negative. After an introductory section on Serre's conjectures, we present an outline of this proof. In addition, we discuss related questions on Hilbert polynomials of bi-graded rings.

An outline of Gabber's proof can be found in Berthelot [1], and a more complete exposition of the proof is given in Hochster [5]. Both of these articles had a strong influence on these notes.

1. THE SERRE MULTIPLICITY CONJECTURES

In [7], Serre introduced a definition of intersection multiplicity for regular local rings and showed that it satisfied many of the properties which should hold for intersection multiplicities. The definition is as follows: let R be a regular local ring of dimension d, and let $X = \mathrm{Spec}(R)$. Let Y and Z be closed subschemes of X defined by ideals $\mathfrak{p}$ and $\mathfrak{q}$ such that $Y \cap Z$ consists only of the closed point of X, or, equivalently, that $R/\mathfrak{p} \otimes R/\mathfrak{q}$ is a module of finite length. (Despite the notation, it is not necessary that $\mathfrak{p}$ and $\mathfrak{q}$ be prime; however, they will usually be assumed to be prime in later sections of the paper.) Then the intersection multiplicity of Y and Z is defined to be

$$\chi(Y,Z) = \chi(R/\mathfrak{p}, R/\mathfrak{q}) = \sum_{i=0}^{d} (-1)^i \,\mathrm{length}(\mathrm{Tor}_i^R(R/\mathfrak{p}, R/\mathfrak{q}))\,.$$

More generally, if M and N are finitely generated R-modules such that $M \otimes N$ is a module of finite length, we define

$$\chi(M, N) = \sum_{i=0}^{d} (-1)^i \operatorname{length}(\operatorname{Tor}_i^R(M, N)).$$

One of the motivations behind this definition is that it can be shown that Bézout's theorem holds if multiplicities are defined in this way; that is, if Y and Z are closed subschemes of projective space meeting in a finite number of points, then the number of points of intersection counted with multiplicities is the product of the degrees of Y and Z.

On the other hand, there were certain properties which are not obviously satisfied and which were left as conjectures. In the form given by Serre [7], the conjectures are as follows: let R be a regular local ring, and let M and N be finitely generated R-modules such that $M \otimes_R N$ has finite length. Then:

- $\dim(M) + \dim(N) \leq \dim(R)$.
- (Non-negativity) $\chi(M, N) \geq 0$.
- $\chi(M, N) > 0$ if and only if $\dim(M) + \dim(N) = \dim(R)$.

Another version of these conjectures is the following:

- $\dim(M) + \dim(N) \leq \dim(R)$.
- (Vanishing) If $\dim(M) + \dim(N) < \dim(R)$, $\chi(M, N) = 0$.
- (Positivity) If $\dim(M) + \dim(N) = \dim(R)$, $\chi(M, N) > 0$.

It is easy to see that the two sets of conjectures are equivalent. Serre proved the first statement in general, and he proved the others for regular rings containing a field by the method of reduction to the diagonal. We will discuss part of this method below. The question was left open for rings of mixed characteristic, and Serre also asked whether a proof existed which did not use reduction to the diagonal.

The vanishing conjecture was proven about ten years ago (Roberts [6], Gillet-Soulé [3]) using K-theoretic methods. The proof in [6] uses the theory of local Chern characters, while that in [3] uses the theory of Adams operations on Grothendieck groups of complexes.

The main topic of these notes is the recent proof of Gabber of the non-negativity conjecture, in the course of which he also gives a new proof of the vanishing conjecture. In addition, we discuss some questions which arise when attempting to extend these ideas to prove the positivity conjecture. First, we recall a spectral sequence argument used by Serre in his proof and which was extended by Gabber to reduce these questions on modules over regular local rings to questions on locally free sheaves on projective space.