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RECENT DEVELOPMENTS ON SERRE'S MULTIPLICITY CONJECTURES: GABBER'S PROOF OF THE NONNEGATIVITY CONJECTURE

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These notes are based on talks given at the Encuentro de Geometría Algebraica y Álgebra Conmutativa in Guanajuato in August 1997. They describe recent developments in the questions on intersection multiplicities, particularly Gabber's recent proof of Serre's conjecture that intersection multiplicities over regular local rings are non-negative. After an introductory section on Serre's conjectures, we present an outline of this proof. In addition, we discuss related questions on Hilbert polynomials of bi-graded rings.

An outline of Gabber's proof can be found in Berthelot [1], and a more complete exposition of the proof is given in Hochster [5]. Both of these articles had a strong influence on these notes.

1. THE SERRE MULTIPLICITY CONJECTURES

In [7], Serre introduced a definition of intersection multiplicity for regular local rings and showed that it satisfied many of the properties which should hold for intersection multiplicities. The definition is as follows: let R be a regular local ring of dimension d, and let $X = \mathrm{Spec}(R)$. Let Y and Z be closed subschemes of X defined by ideals $\mathfrak{p}$ and $\mathfrak{q}$ such that $Y \cap Z$ consists only of the closed point of X, or, equivalently, that $R/\mathfrak{p} \otimes R/\mathfrak{q}$ is a module of finite length. (Despite the notation, it is not necessary that $\mathfrak{p}$ and $\mathfrak{q}$ be prime; however, they will usually be assumed to be prime in later sections of the paper.) Then the intersection multiplicity of Y and Z is defined to be

$$\chi(Y, Z) = \chi(R/\mathfrak{p}, R/\mathfrak{q}) = \sum_{i=0}^{d} (-1)^i \,\mathrm{length}(\mathrm{Tor}_i^R(R/\mathfrak{p}, R/\mathfrak{q}))\,.$$

More generally, if M and N are finitely generated R-modules such that $M \otimes N$ is a module of finite length, we define

$$\chi(M, N) = \sum_{i=0}^{d} (-1)^i \operatorname{length}(\operatorname{Tor}_i^R(M, N)).$$

One of the motivations behind this definition is that it can be shown that Bézout's theorem holds if multiplicities are defined in this way; that is, if Y and Z are closed subschemes of projective space meeting in a finite number of points, then the number of points of intersection counted with multiplicities is the product of the degrees of Y and Z.

On the other hand, there were certain properties which are not obviously satisfied and which were left as conjectures. In the form given by Serre [7], the conjectures are as follows: let R be a regular local ring, and let M and N be finitely generated R-modules such that $M \otimes_R N$ has finite length. Then:

- $\dim(M) + \dim(N) \leq \dim(R)$.
- (Non-negativity) $\chi(M, N) \geq 0$.
- $\chi(M, N) > 0$ if and only if $\dim(M) + \dim(N) = \dim(R)$.

Another version of these conjectures is the following:

- $\dim(M) + \dim(N) \leq \dim(R)$.
- (Vanishing) If $\dim(M) + \dim(N) < \dim(R)$, $\chi(M, N) = 0$.
- (Positivity) If $\dim(M) + \dim(N) = \dim(R)$, $\chi(M, N) > 0$.

It is easy to see that the two sets of conjectures are equivalent. Serre proved the first statement in general, and he proved the others for regular rings containing a field by the method of reduction to the diagonal. We will discuss part of this method below. The question was left open for rings of mixed characteristic, and Serre also asked whether a proof existed which did not use reduction to the diagonal.

The vanishing conjecture was proven about ten years ago (Roberts [6], Gillet-Soulé [3]) using K-theoretic methods. The proof in [6] uses the theory of local Chern characters, while that in [3] uses the theory of Adams operations on Grothendieck groups of complexes.

The main topic of these notes is the recent proof of Gabber of the non-negativity conjecture, in the course of which he also gives a new proof of the vanishing conjecture. In addition, we discuss some questions which arise when attempting to extend these ideas to prove the positivity conjecture. First, we recall a spectral sequence argument used by Serre in his proof and which was extended by Gabber to reduce these questions on modules over regular local rings to questions on locally free sheaves on projective space.

2. The Serre spectral sequence

The main theorem of this section relates the Euler characteristic of a Koszul complex on a module to the Samuel multiplicity of the module. Let A be a local ring, and let M be a finitely generated A-module of dimension at most k. Let $\mathfrak{a}$ be an ideal of A such that $M/\mathfrak{a}M$ has finite length. We recall that the associated Hilbert-Samuel polynomial $P_M^{\mathfrak{a}}(n)$ is defined to be the polynomial for which

$$P_M^{\mathfrak{a}}(n) = \text{length}(M/\mathfrak{a}^n M)$$

for large n. If the dimension of M is at most k, we define the Samuel multiplicity $e_k(\mathfrak{a}, M)$ to be $k!$ times the coefficient of n^k in $P_M^{\mathfrak{a}}(n)$ (if the dimension of M is less than k, $e_k(\mathfrak{a}, M)$ will be zero).

THEOREM 1. *With notation as above, let $x_1, \dots, x_k$ be a sequence of elements of A, and let $\mathfrak{a}$ be the ideal generated by $x_1, \dots, x_k$. Assume that $M/\mathfrak{a}M$ is a module of finite length. Let $K_\bullet$ be the Koszul complex on $x_1, \dots, x_k$, and let*

$$\chi(K_\bullet \otimes M) = \sum_{i=0}^{k} (-1)^i \,\text{length}(H_i(K_\bullet \otimes M)) .$$

Then

$$\chi(K_\bullet \otimes M) = e_k(\mathfrak{a}, M) .$$

We sketch the argument used to prove this theorem. The main idea is to examine the spectral sequence defined by the filtration on $K_\bullet$ induced by powers of $\mathfrak{a}$. For each $n \geq 0$ and for each i we consider the quotient $\mathfrak{a}^n K_i / \mathfrak{a}^{n+1} K_i$. For each $r \geq 0$ we then take the subquotient $E^r_{i,n}$ of this module defined by

$$E^r_{i,n} = \frac{\{k_i \in \mathfrak{a}^n K_i \mid d_i(k_i) \in \mathfrak{a}^{n+r} K_{i-1}\} + \mathfrak{a}^{n+1} K_i}{(\{d_{i+1}(k_{i+1}) \mid k_{i+1} \in \mathfrak{a}^{n-r+1} K_{i+1}\} \cap \mathfrak{a}^n K_i) + \mathfrak{a}^{n+1} K_i} .$$

The $E^r_{i,n}$ define a spectral sequence (the usual spectral sequence associated to a filtered complex). While the precise definition is necessarily quite complicated, the idea is that $E^r_{i,n}$ is the subquotient of $\mathfrak{a}^n K_i / \mathfrak{a}^{n+1} K_i$ consisting of elements whose boundaries lie r steps further down in the filtration modulo boundaries of elements which lie at most $r-1$ steps further up in the filtration. As r gets large, this subquotient approaches the submodule of elements whose boundaries are zero modulo the submodule consisting of all of the boundaries.

In fact, it can be shown using the Artin-Rees lemma (see Serre [7]) that the spectral sequence does in fact converge to the $\mathfrak{a}$-adic filtration on the homology of $K_\bullet$.

Part of the general theory of spectral sequences, which can be verified directly in this case from the above definition, is that the boundary map d_i on $K_\bullet$ induces a map $d^r_{i,n}$ from $E^r_{i,n}$ to $E^r_{i-1,n+r}$ for each i, n and r, and that we have

$$E^{r+1}_{i,n} = \operatorname{Ker}(d^r_{i,n})/\operatorname{Im}(d^r_{i+1,n-r}).$$

Thus the modules at stage $r+1$ can be computed as the homology of those at the r^{th} stage under maps induced by the boundary maps of $K_\bullet$.

We next examine the complexes defined by $E^0_{i,n}$ and $E^1_{i,n}$.

If we let $r = 0$ in the above definition of $E^r_{i,n}$, the condition that $d_i(k_i) \in \mathfrak{a}^{n+r}K_{i-1}$ states that $d_i(k_i) \in \mathfrak{a}^n K_{i-1}$, which is always true since k_i is assumed to be in $\mathfrak{a}^n K_i$ and d_i is a module homomorphism. Similarly, the condition that $k_{i+1} \in \mathfrak{a}^{n+1}K_{i+1}$ implies that $d_{i+1}(k_{i+1}) \in \mathfrak{a}^{n+1}K_i$, so that when $r = 0$ the denominator in the above definition of $E^r_{i,n}$ is just $\mathfrak{a}^{n+1}K_i$. Hence $E^0_{i,n}$ is simply $\mathfrak{a}^n K_i/\mathfrak{a}^{n+1}K_i$. Furthermore, since $K_\bullet$ is the Koszul complex on the generators of $\mathfrak{a}$, the maps d_i are all zero modulo $\mathfrak{a}$, and the maps induced by the boundary maps d_i on $E^0_{i,n}$ are zero. It then follows that $E^1_{i,n}$ is also equal to $\mathfrak{a}^n K_i/\mathfrak{a}^{n+1}K_i$.

We next consider the maps $d^1_{i,n}$ induced by d_i on $E^1_{i,n}$; we denote this map $\bar{d}_i$. Since $K_\bullet$ is the Koszul complex on $x_1, \dots, x_k$, the map d_i is defined by a matrix with $\pm x_i$ in certain positions and zeros in the remaining positions. Thus $\bar{d}_i$ is defined by the same matrix in which x_i is considered as a map from $\mathfrak{a}^n K_i/\mathfrak{a}^{n+1}K_i$ to $\mathfrak{a}^{n+1}K_{i-1}/\mathfrak{a}^{n+2}K_{i-1}$ for each n. Let $\overline{K}_i$ denote the associated graded module of K_i under the filtration by powers of $\mathfrak{a}$. Then $\bar{d}_i$ defines a map of degree one from $\overline{K}_i$ to $\overline{K}_{i-1}$, and the above description shows that the resulting complex is the Koszul complex on $\bar{x}_1, \dots, \bar{x}_k$, where $\bar{x}_i$ denotes the image of x_i in the component of degree 1 of the associated graded ring $gr_{\mathfrak{a}}(A)$.

Thus we have shown that if for each i we let

$$\overline{K}_i = \bigoplus_{n \geq 0} E^1_{i,n} = \bigoplus_{n \geq 0} \mathfrak{a}^n K_i/\mathfrak{a}^{n+1}K_i,$$

the maps $d^1_{i,n}$ induced by d_i define a complex $\overline{K}_\bullet$ which is the Koszul complex on $\bar{x}_1, \dots, \bar{x}_k$ over $gr_{\mathfrak{a}}(A)$.

Up to now we have considered the filtration on $K_\bullet$ without mentioning the module M. However, exactly the same argument holds for $K_\bullet \otimes M$, and we obtain a spectral sequence $E^r_{i,n}(M)$ which converges to the homology of

$K_\bullet \otimes M$ and such that the modules $E^1_{i,n}(M)$ form the Koszul complex induced by $\bar{x}_1, \ldots, \bar{x}_k$ on the associated graded module $gr_{\mathfrak{a}}(M)$. This Koszul complex can also be expressed as $\bar{K}_\bullet \otimes gr_{\mathfrak{a}}(M)$. Since we are assuming that $M/\mathfrak{a}M$ has finite length, the homology of the Koszul complex induced by $\bar{x}_1, \ldots, \bar{x}_k$ on $gr_{\mathfrak{a}}(M)$ also has finite length. Thus, since stage $r+1$ of the spectral sequence is obtained from the r^{th} stage by taking homology, the Euler characteristic is preserved from each stage of the spectral sequence to the next. Hence, since the spectral sequence converges to the homology of $K_\bullet \otimes M$, we have

$$\chi(K_\bullet \otimes_A M) = \chi\big(\bar{K}_\bullet \otimes_{gr_{\mathfrak{a}}(A)} gr_{\mathfrak{a}}(M)\big).$$

To complete the proof that this Euler characteristic is equal to the Samuel multiplicity, we interpret the complex $\bar{K}_\bullet \otimes_{gr_{\mathfrak{a}}(A)} gr_{\mathfrak{a}}(M)$ as a complex of graded modules. Denote this complex $\bar{K}^M_\bullet$. Each module $\bar{K}^M_i$ has a Hilbert polynomial P_i such that

$$P_i(n) = \sum_{j=0}^{n-1} \operatorname{length}(\bar{K}^M_i)_j,$$

where $(\bar{K}^M_i)_j$ denotes the component of $\bar{K}^M_i$ of degree j. However, since $\bar{K}^M_\bullet$ is a Koszul complex on the associated graded module of M, we also have

$$P_i(n) = \binom{k}{i} P^{\mathfrak{a}}_M(n-i)$$

for all i, where $P^{\mathfrak{a}}_M$ is the Hilbert polynomial of M. The shift by i in $\bar{K}^M_\bullet$ is necessary so that the boundary maps will be maps of graded modules of degree zero. By the additivity of Hilbert polynomials, $\sum_{i=0}^k (-1)^i P_i(n)$ gives the Hilbert polynomial defined by the homology of $\bar{K}^M_\bullet$, which is constant with value $\chi(\bar{K}^M_\bullet)$. But a direct computation (we prove a more general version of this in a later section) shows that $\sum_{i=0}^k (-1)^i \binom{k}{i} P^{\mathfrak{a}}_M(n-i)$ is $k!$ times the coefficient of n^k in $P^{\mathfrak{a}}_M(n)$, which proves the result.

The point of this computation is that it transforms questions about Euler characteristics into questions about Hilbert polynomials, which are often easier to deal with. We consider one particularly important case. Let R be a regular local ring, and let $\mathfrak{p}$ and $\mathfrak{q}$ be ideals of R such that $R/\mathfrak{p} \otimes R/\mathfrak{q}$ has finite length. Suppose $\mathfrak{q}$ is generated by a regular sequence $x_1, \ldots, x_k$. Then $\dim(R/\mathfrak{q}) = \dim(R) - k$, so that we have $\dim(R/\mathfrak{p}) \le \dim(R) - \dim(R/\mathfrak{q}) = k$, and $\dim(R/\mathfrak{p}) + \dim(R/\mathfrak{q}) = \dim(R)$ if and only if $\dim(R/\mathfrak{p}) = k$. Since $x_1, \ldots, x_k$ is a regular sequence, the Koszul complex $K_\bullet$ on $x_1, \ldots, x_k$ is a free resolution of $R/\mathfrak{q}$. Thus $\mathrm{Tor}_i^R(R/\mathfrak{q}, R/\mathfrak{p})$ is the homology $H_i(K_\bullet \otimes R/\mathfrak{p})$. Applying the above theorem with $M = R/\mathfrak{p}$, we deduce that

$$\chi(R/\mathfrak{q}, R/\mathfrak{p}) = e_k(\mathfrak{q}, R/\mathfrak{p}).$$

Since the Samuel multiplicity $e_k(\mathfrak{q}, R/\mathfrak{p})$ is always non-negative and is positive if and only if the dimension of $R/\mathfrak{p}$ is equal to k, this proves the conjectures in this case.

Serre's proof of the multiplicity conjectures in the equicharacteristic case proceeded by reducing to the case of a regular sequence by reduction to the diagonal. If R is a power series ring $k[[X_1, \ldots, X_d]]$ and M and N are R-modules with $M \otimes_R N$ of finite length, he introduced a new set of variables $Y_1, \ldots, Y_d$ and considered N as a module over $k[[Y_1, \ldots, Y_d]]$. He then defined a "complete" tensor product $M \widehat{\otimes}_k N$ over k as a module over the ring $k[[X_1, \ldots, X_d, Y_1, \ldots, Y_d]]$ and showed that

$$\mathrm{Tor}_i^R(M, N) \cong \mathrm{Tor}_i^{k[[X_i, Y_j]]}(M \widehat{\otimes}_k N, k[[X_i, Y_j]]/(X_1 - Y_1, \ldots, X_d - Y_d)).$$

Since $X_1 - Y_1, \ldots, X_d - Y_d$ form a regular sequence, this proves the result for power series rings, and the conjectures for general equicharacteristic rings can be reduced to this case by completion and the Cohen structure theorems.

3. Gabber's reduction to regular embeddings

In this section we describe Gabber's use of de Jong's theorem on the existence of "regular alterations" to reduce the intersection conjectures to questions on regular embeddings in projective space over R.

As above, let R be a regular local ring and let $\mathfrak{p}$ and $\mathfrak{q}$ be prime ideals of R such that $R/\mathfrak{p} \otimes R/\mathfrak{q}$ has finite length. Let d be the dimension of R, let r be the dimension of $R/\mathfrak{p}$ and let t be the dimension of $R/\mathfrak{q}$.

The following theorem of de Jong [2] makes the reduction to a question on regular embeddings possible:

THEOREM 2. *Let A be a local integral domain which is a localization of a ring of finite type over a discrete valuation ring. Then there exists a projective map $\phi\colon X \to \mathrm{Spec}(A)$ such that*

- *X is an integral regular scheme.*
- *If K is the quotient field of A, then the extension $k(X)$ of K is finite (we say that X is* generically finite *over* $\mathrm{Spec}(A)$).

For the proof of this theorem we refer to [2]. We show below that this reduces the questions on intersections over a regular local ring to corresponding questions on intersections on projective schemes where one of the schemes is regular. We note that the fact that ϕ is projective means that X is a closed subscheme of $\mathrm{Proj}(A[X_0,\dots,X_n])$ for some n. In our application, we apply the theorem to $A = R/\mathfrak{q}$. Suppose first that $\mathrm{Spec}(R/\mathfrak{q})$ is already regular, which means that $R/\mathfrak{q}$ is a regular local ring. In that case, $\mathfrak{q}$ is generated by part of a regular system of parameters, so is in particular generated by a regular sequence. Hence the conjectures follow immediately from the results of the previous section on Koszul complexes.

We note that there is an extra assumption, that the ring be a localization of a ring of finite type over a discrete valuation ring (and there are also assumptions on the discrete valuation ring). However the general case can be reduced to this case (see Berthelot [1] or Hochster [5]), and we assume that our rings have this property.

Let $X = \mathrm{Spec}(R)$, $Z = \mathrm{Spec}(R/\mathfrak{q})$ and $Y = \mathrm{Spec}(R/\mathfrak{p})$. We denote a regular scheme which is projective and generically finite over Z (whose existence follows from de Jong's theorem) by Z'. Then there exists an n such that Z' is a closed subscheme of $\mathrm{Proj}(R/\mathfrak{q}[X_0,\dots,X_n])$ and hence also of $\mathrm{Proj}(R[X_0,\dots,X_n])$. We let P denote $\mathrm{Proj}(R[X_0,\dots,X_n])$ and let ϕ denote both the map from P to X and the induced map from Z' to Z. Let I denote the graded ideal of $R[X_0,\dots,X_n]$ which defines Z'. Let $Y' = \phi^{-1}(Y) = \mathrm{Proj}(R/\mathfrak{p}[X_0,\dots,X_n])$.

The generalization from rings to projective schemes involves a corresponding generalization from modules to sheaves. The sheaves we consider will be coherent (see for example Hartshorne [4] for the general theory of sheaves on projective schemes). We recall that a coherent sheaf $\mathcal{M}$ on W can be defined either by specifying its modules of sections over the open sets in an affine open cover or, alternatively, by taking the sheaf defined by a finitely generated graded A-module M. We will generally use the second definition, as it is usually more convenient in computing examples.

We let $\mathcal{O}_P$, $\mathcal{O}_{Y'}$, and $\mathcal{O}_{Z'}$ denote the structure sheaves of P, Y', and Z' respectively; they are defined by the graded rings $R[X_0,\dots,X_n]$, $R/\mathfrak{p}[X_0,\dots,X_n]$, and $R[X_0,\dots,X_n]/I$. We will also sometimes denote R by $\mathcal{O}_X$ and similarly for $\mathcal{O}_Y$ and $\mathcal{O}_Z$.

If $\mathcal{M}$ and $\mathcal{N}$ are sheaves on a projective scheme W defined by graded modules M and N, we define the sheaves $\mathcal{T}or_i^{\mathcal{O}_W}(\mathcal{M},\mathcal{N})$ by taking a resolution of $\mathcal{M}$ (or $\mathcal{N}$) by locally free sheaves $\mathcal{F}_i$, and letting $\mathcal{T}or_i^{\mathcal{O}_W}(\mathcal{M},\mathcal{N})$ be the i^{th} homology of $\mathcal{F}_\bullet \otimes \mathcal{N}$. Usually we define $\mathcal{F}_\bullet$ by defining a complex of graded modules F_i which define locally free sheaves and which give a resolution of M (or N). In the case where $W = P$, a bounded resolution can be constructed using direct sums of copies of $\mathcal{O}_P(n)$, so this process is quite easy to carry out. We also define the complex $\mathcal{T}or^{\mathcal{O}_W}(\mathcal{M},\mathcal{N})$ to be the complex $\mathcal{F}_\bullet \otimes \mathcal{N}$. This complex is of course not well-defined as a complex, but it is well-defined up to quasi-isomorphism.

The last ingredient in the generalization to projective space is the pushdown of complexes from P to X by the map ϕ, which we denote ϕ_*. In general this functor is the derived functor of the global section functor on sheaves, but in the case of projective space over R it is not difficult to give a direct definition using Cech cohomology. Let $A = R[X_0, \dots, X_n]$, and let $P = \mathrm{Proj}(A)$ as above. Let $C^\bullet$ be the complex

$$0 \to \prod A_{X_i} \to \prod A_{X_iX_j} \to \cdots \to A_{X_0X_1\cdots X_n} \to 0$$

where for any element $Y \in A$, A_Y denotes the localization of A obtained by inverting Y. If $\mathcal{M}_\bullet$ is a bounded complex of coherent sheaves over P represented by a complex of graded modules $M_\bullet$, we then define $\phi_*(\mathcal{M}_\bullet)$ to be the graded part of degree zero of the complex $C^\bullet \otimes_A M_\bullet$. Then $\phi_*(\mathcal{M}_\bullet)$ is a bounded complex of R-modules with finitely generated homology and is well-defined up to quasi-isomorphism.

Now suppose that $\mathcal{M}$ and $\mathcal{N}$ are coherent sheaves on P such that $\mathcal{M} \otimes_{\mathcal{O}_P} \mathcal{N}$ has support which lies over the closed point of R, which we denote s. Then $\mathcal{T}or_i^{\mathcal{O}_P}(\mathcal{M},\mathcal{N})$ has support lying over s for all i, so that the homology of $\phi_*(\mathcal{T}or^{\mathcal{O}_P}(\mathcal{M},\mathcal{N}))$ is supported at the maximal ideal and thus has finite length. Hence we can define

$$\chi(\mathcal{M},\mathcal{N}) = \sum (-1)^i \,\mathrm{length}\left(H^i(\phi_*(\mathcal{T}or^{\mathcal{O}_P}(\mathcal{M},\mathcal{N})))\right).$$

The first part of the reduction is to show that it suffices to show that the Euler characteristic $\chi(\mathcal{O}_{Z'}, \mathcal{O}_{Y'})$ is non-negative and is zero if $\dim(R/\mathfrak{p}) + \dim(R/\mathfrak{q}) < \dim(R)$. The point is that the assumptions on ϕ imply that this new Euler characteristic is closely related to the Euler characteristic $\chi(R/\mathfrak{p}, R/\mathfrak{q})$ defined earlier. Let $G_\bullet$ be a finite free resolution of $R/\mathfrak{p}$ over R. Let $\mathcal{F}_\bullet$ be a finite locally free resolution of $\mathcal{O}_{Z'}$ as above, and let $\mathcal{F}_\bullet$ be defined by a complex $F_\bullet$ of graded modules. We then have the following "projection formula":

$$\phi_*(\mathcal{F}_\bullet) \otimes G_\bullet \cong \phi_*(\mathcal{F}_\bullet \otimes \phi^*(G_\bullet)).$$

To prove this formula, we use the definition of ϕ_* in terms of the complex $C^\bullet$ defined above. The associativity of the tensor product implies that we have isomorphisms of complexes:

$$(C^\bullet \otimes_A F_\bullet) \otimes_R G_\bullet \cong C^\bullet \otimes_A (F_\bullet \otimes_R G_\bullet) \cong C^\bullet \otimes_A (F_\bullet \otimes_A (A \otimes_R G_\bullet)).$$

Since $A \otimes_R G_\bullet$ defines a locally free resolution of $\mathcal{O}_{Y'}$, it is clear that the complex $C^\bullet \otimes_A (F_\bullet \otimes_A (A \otimes_R G_\bullet))$ represents $\phi_*(\mathcal{F}_\bullet \otimes \phi^*(G_\bullet))$. Since $(C^\bullet \otimes_A F_\bullet) \otimes_R G_\bullet$ represents $\phi_*(\mathcal{F}) \otimes G_\bullet$, this proves the above isomorphism.

To complete the proof of the fact that it suffices to prove non-negativity and vanishing for $\chi(\mathcal{O}_{Z'}, \mathcal{O}_{Y'})$, we use induction on the dimension of $R/\mathfrak{q}$ together with the assumption that the map induced by ϕ from Z' to Z is generically finite. Let $\mathcal{F}_\bullet$ be a locally free resolution of $\mathcal{O}_{Z'}$ on P. If we localize at $\mathfrak{q}$, the generic finiteness of ϕ implies that the resulting map from $\mathrm{Proj}((A/I)_\mathfrak{q})$ to $\mathrm{Spec}((R/\mathfrak{q})_\mathfrak{q})$ is defined by a finite field extension of a given degree which we denote n. Thus $\phi_*(\mathcal{F}_\bullet)$ localized at $\mathfrak{q}$ is isomorphic to $((R/\mathfrak{q})_\mathfrak{q})^n$, so the complex $\phi_*(\mathcal{F}_\bullet)$ is isomorphic to the module $(R/\mathfrak{q})^n$ up to a complex with homology of dimension strictly less than the dimension of $R/\mathfrak{q}$.

By the projection formula, we have that

$$\phi_*(\mathcal{F}_\bullet) \otimes G_\bullet \cong \phi_*(\mathcal{F}_\bullet \otimes \phi^*(G_\bullet)),$$

where $G_\bullet$ is a free resolution of $R/\mathfrak{p}$. Since $\mathcal{F}_\bullet$ is a locally free resolution of $\mathcal{O}_{Z'}$ and $\phi^*(G_\bullet)$ is a locally free resolution of $\mathcal{O}_{Y'}$, the complex $\phi_*(\mathcal{F}_\bullet \otimes \phi^*(G_\bullet))$ is quasi-isomorphic to $\phi_*(\mathcal{T}or^{\mathcal{O}_P}(\mathcal{O}_{Z'}, \mathcal{O}_{Y'}))$. Hence, taking Euler characteristics and using the above isomorphism, we have $\chi(\phi_*(\mathcal{F}_\bullet), R/\mathfrak{p}) = \chi(\mathcal{O}_{Z'}, \mathcal{O}_{Y'})$. Applying the induction hypothesis, we have that $\chi(M, R/\mathfrak{p})$ is zero whenever the dimension of M is less than the dimension of $R/\mathfrak{q}$. Thus, since $\phi_*(\mathcal{O}_{Z'})$ is isomorphic to $(R/\mathfrak{q})^n$ up to something of dimension strictly less than the dimension of $R/\mathfrak{q}$, we have that

$$\chi(\mathcal{O}_{Z'}, \mathcal{O}_{Y'}) = \chi(\phi_*(\mathcal{F}_\bullet), R/\mathfrak{p}) = \chi((R/\mathfrak{q})^n, R/\mathfrak{p}) = n(\chi(R/\mathfrak{q}, R/\mathfrak{p})).$$

Thus the vanishing, non-negativity, and positivity of $\chi(\mathcal{O}_{Z'}, \mathcal{O}_{Y'})$ are equivalent to the corresponding properties of $\chi(R/\mathfrak{q}, R/\mathfrak{p})$.

Thus we have reduced the multiplicity conjectures to corresponding conjectures on Euler characteristics defined by subschemes Y' and Z' of projective space over R, where Z' is regular and Y' is the pullback of a subscheme of $\mathrm{Spec}(R)$. In particular, the ideal I defining Z' is locally generated by a regular sequence, and this fact makes it possible to use the Serre spectral sequence to reduce to the case of associated graded rings.

Let $gr_I(A) = A/I \oplus I/I^2 \oplus \dots$ be the associated graded ring of I. Let B denote $A/\mathfrak{p}A$, and let $gr_I(B)$ denote the associated graded ring of the image of I in B. We note that both $gr_I(A)$ and $gr_I(B)$ are bigraded rings, with one grading induced by the grading on A and the other corresponding to powers of I. The ring $gr_I(B)$ is also a bigraded module over $gr_I(A)$. We make the convention that the i,j component of $gr_I(A)$ is the component of I^j/I^{j+1} of degree i. We let E denote the scheme $\mathrm{Proj}(gr_I(A))$, where $gr_I(A)$ is considered to be a graded module by the grading in the first component (the grading induced from that on A). Then E can be defined locally as follows: if U is an affine open set in Z' and $\mathcal{O}_U$ is the ring such that $U = \mathrm{Spec}(\mathcal{O}_U)$, then the fiber of E over U is defined to be $\mathrm{Spec}(C)$, where C is the associated graded ring of $\mathcal{O}_U$ by the restriction of I to U. Since I is locally generated by a regular sequence, C is locally a polynomial ring over $\mathcal{O}_U$. We note that $\mathcal{O}_{Z'}$ is a quotient of both A and $gr_I(A)$. Let $\mathcal{M}$ denote the sheaf on E defined by the graded module $gr_I(B)$.

We next show that the Serre spectral sequence implies that we have an equality:

$$\chi_E(\mathcal{O}_{Z'}, \mathcal{M})) = \chi_P(\mathcal{O}_{Z'}, \mathcal{O}_{Y'}) .$$

Let

$$0 \to \mathcal{F}_k \to \cdots \to \mathcal{F}_1 \to \mathcal{F}_0 \to 0$$

be a locally free resolution of $\mathcal{O}_{Z'}$ over $\mathcal{O}_P$. We apply the argument of section 2 to the filtration of $\mathcal{F}_\bullet$ induced by the powers of I. Since I is locally generated by a regular sequence, the same argument goes through. However, there are two points which are different from the case of the Koszul complex. First of all, $\mathcal{F}_\bullet$ will in general not be a minimal complex locally, so that it is locally a direct sum of a Koszul complex and a trivial (split exact) complex. However, in the local computation, the split exact part is eliminated in the step from E^0 to E^1, so from that point the argument goes through as before. The second point is that in taking the homology at E^1, the homology is no longer of finite length, but only supported at the maximal ideal of R. However, it is still zero except for finitely many i and n and we can conclude that the Euler characteristic is the same using the additivity of ϕ_* and the Euler characteristic on $\mathrm{Spec}(R)$. Thus the argument goes through, and we have the above equality.

There is one more reduction, which reduces to the fibers over $\mathrm{Spec}(R/\mathfrak{m}) = s$. Let $\mathcal{M}$ be the sheaf on E with associated graded ring $gr_I(B)$ considered as a module over $gr_I(A)$. Then, since for the original ideals $\mathfrak{p}$ and $\mathfrak{q}$ we had that $R/\mathfrak{p} \otimes R/\mathfrak{q}$ had finite length, $\mathcal{M}$ is annihilated by a power of the maximal ideal $\mathfrak{m}$ of R. Hence it has a finite filtration with quotients $\mathcal{M}_i$ which are annihilated by $\mathfrak{m}$. It then suffices to show that $\chi_E(\mathcal{O}_{Z'}, \mathcal{M}_i)$ is non-negative for each i. We can compute this Euler characteristic by taking a locally free resolution for $\mathcal{O}_{Z'}$ and tensoring with $\mathcal{M}_i$, and, since $\mathcal{M}_i$ is annihilated by $\mathfrak{m}$, we can tensor first with $R/\mathfrak{m}$. Let s denote the closed point of Spec(R) as above, let $E_s = \text{Proj}(gr_I(A) \otimes_R k)$, where $k = R/\mathfrak{m}$, and let $Z'_s = \text{Proj}(A/I \otimes_R k)$. The above argument shows that for each i we have $\chi_E(\mathcal{O}_{Z'}, \mathcal{M}_i) = \chi_{E_s}(\mathcal{O}_{Z'_s}, \mathcal{M}_i)$. Hence

$$\chi_E(\mathcal{O}_{Z'}, \mathcal{M}) = \sum_i \chi_E(\mathcal{O}_{Z'}, \mathcal{M}_i) = \sum_i \chi_{E_s}(\mathcal{O}_{Z'_s}, \mathcal{M}_i).$$

We recall that the dimension of $\mathcal{M}$ is equal to $\dim(R/\mathfrak{p}) + n$ (where $P = \text{Proj}(R[X_0, \ldots, X_n])$). Thus to prove the vanishing and non-negativity conjectures it suffices to show that whenever $\mathcal{M}$ is a coherent sheaf on E_s and $\dim(\mathcal{M}) + \dim(Z') \leq \dim(R) + n$ we have $\chi_{E_s}(\mathcal{O}_{Z'_s}, \mathcal{M}) \geq 0$, and that we have equality when $\dim(\mathcal{M}) + \dim(Z') < \dim(R) + n$.

To prove the positivity conjecture it would of course suffice to show that if $\dim(\mathcal{M}) + \dim(Z') = \dim(R) + n$, the Euler characteristic is positive. However, this is not true in general (we give an example below). However, assuming the non-negativity conjecture for a moment, we show that there is a simple criterion for positivity.

PROPOSITION 1. *Let notation be as above, and let $\mathcal{M}_0$ be the sheaf defined by $gr_I(B) \otimes_R k$ considered as a module over $gr_I(A) \otimes_R k$. Assume that* $\dim(R/\mathfrak{p}) + \dim(R/\mathfrak{q}) = \dim(R)$. *Then the positivity conjecture holds for the ideals $\mathfrak{p}$ and $\mathfrak{q}$ if and only if $\chi_{E_s}(\mathcal{O}_{Z'_s}, \mathcal{M}_0) > 0$.*

Proof. Since $\mathcal{M}_0$ is a quotient of the sheaf $\mathcal{M}$ defined by $gr_I(B)$ and Euler characteristics are non-negative, if $\chi_{E_s}(\mathcal{O}_{Z'_s}, \mathcal{M}_0) > 0$, then $\chi_E(\mathcal{O}_{Z'}, \mathcal{M}) > 0$. Conversely, suppose that $\chi_{E_s}(\mathcal{O}_{Z'_s}, \mathcal{M}_0) = 0$. Since $gr_I(B)$ is annihilated by a power of $\mathfrak{m}$, it has a filtration with quotients which are homomorphic images of direct sums of copies of $gr_I(B) \otimes_R k$. Again using non-negativity, we can deduce that if $\mathcal{M}_i$ is the sheaf defined by any of these quotients, then $\mathcal{M}_i$ is a quotient of a direct sum of copies of $\mathcal{M}_0$, so we have $\chi_{E_s}(\mathcal{O}_{Z'_s}, \mathcal{M}_i) = 0$. Thus the additivity of the Euler characteristic implies that $\chi_E(\mathcal{O}_{Z'}, \mathcal{M}) = 0$.

4. Some examples

We give here three simple examples of the type of bigraded ring which might result from this construction. Each of these examples is obtained by taking a birational map from a regular scheme to $\mathrm{Spec}(R/\mathfrak{q})$, and the last two are simple resolutions of singularities.

We first summarize the construction up to this point. We began with a regular local ring R and prime ideals $\mathfrak{p}$ and $\mathfrak{q}$. We then took a regular subscheme Z' of $\mathrm{Proj}(R[X_0,\ldots,X_n])$ which was generically finite over $\mathrm{Spec}(R/\mathfrak{q})$. The next step was to replace $R[X_0,\ldots,X_n]$ with the associated graded ring of I tensored with $R/\mathfrak{m} = k$. The sheaf $\mathcal{O}_{Y'}$ defined by $B = R/\mathfrak{p}[X_0,\ldots,X_n]$ was then replaced with the sheaf $\mathcal{M}$ defined by $gr_I(B)$, again tensored with k. The assumption of regularity implies that I/I^2 is locally free over A/I; denote its rank r. Then the dimension of $\mathcal{M}$ is at most r, and it is equal to r if and only if we had $\dim(R/\mathfrak{p}) + \dim(R/\mathfrak{q}) = \dim(R)$. We note that the fiber Z'_s of Z' over the maximal ideal of R has dimension at most $\dim(R/\mathfrak{q}) - 1$, but apart from that we do not know much about it. It is the projective scheme defined by the graded ring $(A/I) \otimes_R k$, which is the part of degree zero in the grading in the second component of the bigraded ring we are considering.

For the first example, let R have dimension four, let t, u, v, w be a regular system of parameters, and define the prime ideals $\mathfrak{p}$ and $\mathfrak{q}$ by letting $\mathfrak{p} = (t, u)$ and $\mathfrak{q} = (v, w)$. In this case, $\mathrm{Spec}(R/\mathfrak{q})$ is already regular, and we can simply take the projective scheme $\mathrm{Proj}(R[X]) = \mathrm{Spec}(R)$.

For a slightly more complicated example, consider the subscheme of the projective space $\mathrm{Proj}(R[X, Y])$ defined by the ideal I generated by v, w, and $Xu - Yt$. Then Z' is the blow up of $R/\mathfrak{q}$ at the point defined by the maximal ideal, and the fiber over s is projective space of dimension 1. One could define similar examples in higher dimension.

For a third example, let R have dimension 2, and let I be generated by $Xu - Yt, Zu - Xt, X^2 - YZ$. The projective space P has dimension 2, and the fiber over the maximal ideal has codimension one in $\mathrm{Proj}(k[X, Y, Z])$ and thus has dimension one. The sheaf defined by I/I^2 has rank 2, but I is minimally generated by three elements.

In the above examples it was not really necessary to reduce to projective space since the original quotients $R/\mathfrak{q}$ were regular. We next consider an example where the original scheme is not regular. Let $\mathfrak{m}$ be minimally generated by t, u, and let $\mathfrak{q}$ be the principal ideal generated by $t^2 - u^3$.

We can resolve the singularity by letting Z' be defined by the ideal in $R[X,Y]$ generated by t^2-u^3, $uX-tY$, X^2-uY^2. The fiber Z'_s in this case is $\mathrm{Proj}(k[X,Y]/(X^2))$.

Finally, we consider the case where $\mathfrak{q}$ is the determinantal ideal in R of dimension 4 generated by $wu-t^2$, $wv-tu$, and $tv-u^2$. In this case the resolution can be found by taking the ideal I in $R[X,Y,Z,W]$ generated by the following elements:

$$Z^2-YW, YZ-XW, Y^2-XZ, uW-vZ, uZ-vY, uY-vX, u^2-tv,$$
$$tW-vY, tZ-vX, tY-uX, tu-wv, t^2-wu, wW-vX, wZ-uX, wY-tX.$$

The fiber over the maximal ideal is a determinantal subvariety of dimension 1.

In a later section we will return to these examples and consider the question of computing the Euler characteristics $\chi_{E_s}(\mathcal{O}_{Z'_s}, \mathcal{M})$ for sheaves $\mathcal{M}$ defined as above by certain prime ideals $\mathfrak{p}$ of R.

5. Hilbert polynomials of bigraded modules

In section 2 we showed how the Serre spectral sequence can be used to express the Euler characteristic defined by a Koszul complex in terms of the Samuel multiplicity. In this section we show that similar results hold in the present situation. We now let C denote the bigraded ring which we previously denoted $gr_I(A)\otimes_R k$, where $C_{i,j}$ consists of the elements of $(I^j/I^{j+1})\otimes k$ of degree i. Thus in our present notation, $E_s=\mathrm{Proj}(C)$, where the grading on C is that in the first coordinate. Let C_0 denote the subring $\bigoplus_i(C_{i,0})$. Let r be the rank of I/I^2, and let M be a bigraded module defining a sheaf $\mathcal{M}$ on E_s of dimension at most r; we define the dimension of M to be the dimension of the associated sheaf. We consider the question of computing the Euler characteristic $\chi_{E_s}(\mathcal{O}_{Z'_s}, \mathcal{M})$, which we also denote $\chi(C_0, M)$.

Let

$$0\to F_k\to\cdots\to F_1\to F_0\to C_0\to 0$$

be a complex of bigraded modules which defines a locally free resolution of C_0 over C. For any finitely generated bigraded module N, we let $P_N(m,n)$ by the Hilbert polynomial of N; more precisely, we define P_N to be the polynomial in two variables such that

$$P_N(m,n)=\sum_{i=0}^{n-1}\mathrm{length}(N_{m,i})$$

for large m and n. The degree of P_N is equal to the dimension of N (that is, the dimension of the sheaf defined by N on E_s). Let M be a bigraded module of dimension at most r as above. Then $M \otimes F_i$ has dimension at most r, and we have that the alternating sum of $P_{M\otimes F_i}$ is constant with value equal to $\chi(C_0, M)$.

We will prove this in a special case below (and reduce the non-negativity conjecture to this special case in the next section). We first briefly consider the question of constructing a resolution $F_\bullet$ of C_0. One method is to take the E^1 term of the Serre spectral sequence as defined in the previous section, starting from a locally free resolution of A/I over A. However, even though A/I has a nice resolution by sums of shifts of A, the resulting locally free sheaves in the resolution over the associated graded ring will not be so simple. An alternative approach is to take a global Kozsul complex

$$\cdots \ \Lambda^2(I/I^2) \otimes gr_I(A) \to \Lambda^1(I/I^2) \otimes gr_I(A) \to gr_I(A) \to A/I \to 0\,.$$

The resolution over C can then be obtained in either of these constructions by tensoring with k. This resolution gives an expression for the Euler characteristic in terms of the Chern classes of I/I^2, but again it is not easy to see how to use this information to compute Euler characteristics.

For the remainder of this section we assume that $I/I^2 \otimes_R k$ is a sum of copies of $\mathcal{O}_{Z'_s}(-k_i)$ for various k_i, so that C is a polynomial ring $C_0[T_1, \dots, T_r]$ over C_0, where T_i has degree $(k_i, 1)$ in the bigrading on C. As mentioned above, the non-negativity conjecture will be reduced to this situation in the next section. In this case the resolution is the usual Koszul complex on $T_1, \dots, T_r$, and the Hilbert polynomial of $M \otimes F_i$ is a sum of Hilbert polynomials of M with shifts in the degrees. Furthermore, the Koszul complex on $T_1, \dots, T_r$ is a tensor product of Koszul complexes on the individual T_i, and we can compute the Hilbert polynomial of the tensor product $K_\bullet(T_1, \dots, T_r) \otimes M$ by tensoring by each factor $K_\bullet(T_i)$ in turn and keeping track of the result. As above, assume that the dimension of M is at most r, and let $Q^r_M(m,n)$ be the component of $P_M(m,n)$ of degree r. Let T_i have degree $(k,1)$, and consider the Hilbert polynomial obtained by tensoring with the complex

$$0 \to C[(-k,-1)] \xrightarrow{T_i} C \to 0\,.$$

The Hilbert polynomial of the resulting complex $K_\bullet(T_i) \otimes M$ will be given by the polynomial whose value at (m,n) is $P_M(m,n) - P_M(m-k, n-1)$. We compute this difference for a monomial $m^i n^j$ and obtain

$$m^i n^j - (m-k)^i (n-1)^j = m^i n^j - (m^i - ikm^{i-1} + \dots)(n^j - jn^{j-1} + \dots)$$
$$= m^i n^j - m^i n^j + ikm^{i-1} n^j + jm^i n^{j-1} + \cdots = ikm^{i-1} n^j + jm^i n^{j-1} + \dots,$$

where the remaining terms have lower degree. Since we are concerned with the component of highest degree, this suffices for our computation. We note that we can express this result by the formula

$$Q^{r-1}_{K_\bullet(T_i)\otimes M} = \frac{\partial Q^r}{\partial n} + k\frac{\partial Q^r}{\partial m}.$$

Iterating this process, where we let T_i have degree $(k_i, 1)$ for each i, we have

$$\chi(C_0, M) = \prod_{i=1}^{r}\left(\frac{\partial}{\partial n} + k_i\frac{\partial}{\partial m}\right) Q^r_M.$$

In this formula Q^r_M could be replaced with P_M.

THEOREM 3. *Let* $C = C_0[T_1, \ldots, T_r]$, *where* T_i *has degree* $(k_i, 1)$ *as above, and let* M *be a bigraded* C*-module of dimension at most* r.

(i) *If* $\dim(M) < r$, *then* $\chi(C_0, M) = 0$.

(ii) *If* $k_i \geq 0$ *for all* i, *then* $\chi(C_0, M) \geq 0$.

(iii) *If* $k_i = 0$ *for all* i, *then* $\chi(C_0, M) > 0$ *if and only if the coefficient of* n^r *in* P_M *is non-zero.*

(iv) *If* $k_i > 0$ *for all* i *and* $\dim(M) = r$, *then* $\chi(C_0, M) > 0$.

Proof. If the dimension of M is less than r, its Hilbert polynomial has degree less that r, so the result of taking r partial derivatives is zero. Thus (i) holds.

We prove (ii) and (iv) by induction on r. By taking a filtration of M, we may assume that M is of the form $(C/\mathfrak{p})[(i,j)]$, where $\mathfrak{p}$ is a bigraded prime ideal of C and $[(i,j)]$ denotes a shift in degrees. Suppose some T_i is not in $\mathfrak{p}$. Then T_i is a non-zero divisor on M, and we can tensor with the Koszul complex on T_i, replacing M with M/T_iM and reducing r by one. Thus the result follows by induction. If all T_i are in $\mathfrak{p}$, then its Hilbert polynomial is constant with respect to n, so we have $Q^r(m,n) = \alpha m^r$ for some $\alpha \geq 0$. Hence the above formula states that

$$\chi(C_0, M) = k_1 k_2 \cdots k_r (r!)\alpha.$$

If all the k_i are greater than or equal to zero, we thus have $\chi(C_0, M) \geq 0$. If all the k_i are greater than zero and M has dimension r, then $\alpha > 0$ and $\chi(C_0, M) > 0$. This proves (ii) and (iv).

If all the k_i are zero, then $\chi(C_0, M)$ is simply the r^{th} derivative of P_M, so it is positive if and only if the coefficient of n^r is positive. On the other hand, this coefficient gives the length of the module $\bigoplus_{i=1}^{n-1} M_{m,i}$ for sufficiently large n up to terms of lower degree in n, so it cannot be negative.

The graded ring obtained from the original situation will be of the form considered here when I is globally defined by a regular sequence, and the k_i will then be the degrees of the generators. We give an example to show that the condition that M has dimension r does not suffice for $\chi(C_0, M)$ to be positive. Let R have dimension 3 and let t, u, v be a regular system of parameters. Let $\mathfrak{q}$ be the ideal generated by v, and let I be the ideal of $R[X, Y]$ generated by v and $uX - tY$. Then the fiber over the closed point is projective space of dimension one, $C_0 = k[X, Y]$, and $C = C_0[T_1, T_2]$ with $k_1 = 0$ and $k_2 = 1$. Then if $M = C/T_1$, M has dimension 2 and $\chi(C_0, M) = 0$.

EXERCISE. Prove (without using the Serre positivity conjecture) that the module M in the previous paragraph could not arise from a prime ideal $\mathfrak{p}$ such that $R/\mathfrak{p} \otimes R/\mathfrak{q}$ has finite length and $\dim(R/\mathfrak{p}) + \dim(R/\mathfrak{q}) = \dim(R)$.

6. GABBER'S PROOF OF NON-NEGATIVITY

In this section we complete Gabber's proof of the non-negativity of intersection multiplicities. We have seen in the last section that if $gr_I(A) \otimes_R k$ is a polynomial ring over $(A/I) \otimes_R k$ generated by elements of non-negative degree, then non-negativity follows. We show here that we can embed $gr_I(A) \otimes_R k$ into a polynomial ring of this type. Let A_0 denote $A/I \otimes_R k$. Actually, we show instead that we can embed the symmetric algebra $\mathrm{Sym}_{A_0}((I/I^2) \otimes_R k)$ into a polynomial ring by a locally flat map. Since I/I^2 is locally free, the map from the symmetric algebra to the associated graded algebra defines an isomorphism of schemes, so this suffices to prove the result. Let $S = \mathrm{Sym}_{A_0}((I/I^2) \otimes_R k)$.

Let E_s denote $\mathrm{Proj}(gr_I(A) \otimes_R k) = \mathrm{Proj}(\mathrm{Sym}_{A_0}((I/I^2) \otimes_R k))$ as above. Let $W = \mathrm{Proj}(A_0[T_1, \ldots, T_{r'}])$ for T_i of degree $(k_i, 1)$ for some integer r'. Suppose that f is a map from S into the polynomial ring $A_0[T_1, \ldots, T_{r'}]$ such that the map ϕ induced by f from W to E_s is flat of relative dimension $r' - r$, where r is the rank of I/I^2. Then we have a commutative diagram

$$\begin{array}{ccc} S & \xrightarrow{\phi} & A_0[T_1, \ldots, T_{r'}] \\ & \searrow \quad \swarrow & \\ & A_0 & \end{array}$$

which induces a commutative diagram of schemes

$$\begin{array}{ccc} E_s & \xleftarrow{\ \phi\ } & W \\ & \nwarrow \quad \nearrow & \\ & Z'_s & \end{array}$$

Let $\mathcal{M}$ be a sheaf on E_s defined by a bigraded module M. Since we are assuming that ϕ is flat, we have an isomorphism

$$\mathcal{T}or_i^{\mathcal{O}_{E_s}}(\mathcal{O}_{Z'_s}, \mathcal{M}) \cong \mathcal{T}or_i^{\mathcal{O}_W}(\mathcal{O}_{Z'_s}, \phi^*(\mathcal{M}))$$

for all i. Thus we have an equality of Euler characteristics

$$\chi_{E_s}(\mathcal{O}_{Z'_s}, \mathcal{M}) = \chi_W(\mathcal{O}_{Z'_s}, \phi^*(\mathcal{M})).$$

Thus if we can find such a map f such that all the k_i are non-negative, the conjecture will follow. We now show that such an embedding exists. Gabber's proof uses the fact that the dual of I/I^2 over s is generated by global sections; we define this map directly without dualizing. At this point we assume that R is ramified. Although this is an unusual assumption, it is possible to reduce to the ramified case by a finite flat extension of R, for example by adjoining a square root of p, where R has mixed characteristic p. Let $t_1, \dots, t_d$ be a minimal set of generators of the maximal ideal $\mathfrak{m}$ of R. Since R is ramified, $R/\mathfrak{m}^2$ is isomorphic to a polynomial ring in $t_1, \dots, t_d$ modulo the square of the ideal generated by $t_1, \dots, t_d$. Thus for each i, the partial derivative $\frac{\partial}{\partial t_i}$ defines a map from $R/\mathfrak{m}^2[X_0, \dots, X_n]$ to $R/\mathfrak{m}[X_0, \dots, X_n]$. By taking the composition with the map from I to $R/\mathfrak{m}^2[X_0, \dots, X_n]$ induced by the inclusion of I into A and with the map from $R/\mathfrak{m}[X_0, \dots, X_n] = A \otimes_R k$ to $(A/I) \otimes_R k = A_0$, we obtain a map from I to A_0. Since for all a in A and i in A the partial derivatives satisfy

$$\frac{\partial(ai)}{\partial t_i} = a\frac{\partial i}{\partial t_i} + i\frac{\partial a}{\partial t_i}$$

and A_0 is annihilated by I, we can deduce that the map induced by $\frac{\partial}{\partial t_i}$ vanishes on I^2 and defines a homomorphism of A/I-modules from I/I^2 to A_0.

Similarly, for each $i = 0, \dots, n$ we have a map induced by $\frac{\partial}{\partial X_i}$ from I/I^2 to $A_0[-1]$, where the shift in degree arises from the fact that these partial derivatives lower the degree by 1.

Putting these together, we have a map from I/I^2 to $A_0^d \oplus A_0[-1]^{n+1}$, which define a map f from S to $A_0[T_1, \dots, T_d, S_0, \dots, S_n]$ where the T_i have degree 0 and the S_i have degree 1.

THEOREM 4. *The map ϕ is locally an inclusion of polynomial rings. In particular, it is locally flat of relative dimension $d+n+1-r$.*

Proof. This is one of the main points of the proof, and it is the only place where the full strength of the assumption that Z' is regular is used. It suffices to show that for every closed point p of Z'_s, the map from I/I^2 to $A_0^d \oplus A_0[-1]^{n+1}$ defines a split inclusion locally at the point p. We assume that the residue field is algebraically closed (which we can do by a flat extension) and look at the maximal ideal $\mathfrak{m}_p$ corresponding to p. The local ring at p in P, which we denote A_p, is isomorphic to $R[u_1, \dots, u_n]_{\mathfrak{m}_p}$, where, after a change of coordinates, we may assume that $u_1, \dots, u_n$ together with a set of generators of $\mathfrak{m}_R$ generate $\mathfrak{m}_p$. Since Z' is regular, I is generated locally by part of a regular system of parameters $i_1, \dots, i_r$. Furthermore, the quotient I/I^2 is locally generated by the images of $i_1, \dots, i_r$. Since $i_1, \dots, i_r$ form part of a regular system of parameters, the images of their partial derivatives in $(A_p/\mathfrak{m}_p)^d \oplus (A_p/\mathfrak{m}_p)^{n+1}$ are linearly independent. Hence the map from $(I/I^2) \otimes k$ to $A_0^d \oplus A_0[-1]^{n+1}$ locally defines a split inclusion at p as was to be shown.

This completes the proof of the Serre nonnegativity conjecture. Since certain of the indeterminates in the polynomial ring used in the proof have degree zero, it does not show that the Euler characteristics must be positive. In fact, as we showed at the end of the previous section, the locally free sheaf defined by I/I^2 is not itself positive enough to ensure positivity. Thus the positivity conjecture requires studying the sheaf $\mathcal{M}$ coming from the associated graded ring of I on $R/\mathfrak{p}[X_0, \dots, X_n]$.

We note that we can embed $A[-1]$ into A^{n+1} by a locally split embedding which sends 1 to $(X_0, \dots, X_n)$ and thus embed S into a polynomial ring D generated by $d+(n+1)^2$ elements all of which have degree zero. Thus one criterion for the positivity conjecture to hold is that if we take the quotient of D by the image in D of the kernel of the map from $gr_I(A)$ to $gr_I(R/\mathfrak{p}[X_0, \dots, X_n])$, then (under the usual assumptions) the coefficient of $n^{d+(n+1)^2}$ in the Hilbert polynomial of this quotient is not zero.

We remark that the construction we have presented is quite computational in the sense that it is possible to compute the embedding ϕ explicitly in special cases. We give two simple examples. First, let R have dimension 2 with maximal ideal generated by t, u, let $A = R[X, Y]$, and let I be generated by $uX - tY$. Then $A_0 = k[X, Y]$. The map f to $A_0[S_1, S_2, T_0, T_1]$ induced by the partial derivatives sends $uX - tY$ to $-YS_1 + XS_2 + uT_0 - tT_1$, which, after dividing by $\mathfrak{m}$, is $-YS_1 + XS_2$. Let $\mathfrak{p} = (t, u)$. Then $uX - tY$ is zero modulo $\mathfrak{p}$, so the kernel on the map of graded rings is generated by the image of $uX - tY$ in I/I^2. Hence $\mathcal{M}$ is mapped to the sheaf associated to $A_0[S_1, S_2, T_0, T_1]/(-YS_1 + XS_2)$. It can be verified that this quotient satisfies the condition on Hilbert polynomials; the positivity condition also follows from the fact that $-YS_1 + XS_2$ has degree $(1, 1)$.

Finally, we consider the example from section 3 in which I is generated by $t^2 - u^3, uX - tY, X^2 - uY^2$. Then I/I^2 has rank 2. Taking derivatives, we see that the map ϕ (after dividing by $\mathfrak{m}$) satisfies $\phi(t^2 - u^3) = 0$, $\phi(uX - tY) = XS_1 - TS_2$, and $\phi(X^2 - uY^2) = -Y^2S_2 + 2XT_0$. To compute the result of intersecting with Y', where Y' is generated by an ideal $\mathfrak{p}$, it suffices to compute the kernel of the map from the symmetric algebra on I/I^2 to the associated graded ring of I on $R/\mathfrak{p}[X, Y]$ tensored with k, and then find the image of this kernel in $A_0[S_1, S_2, T_0, T_1]$. On the other hand, in this case $\mathrm{Proj}(A_0) = \mathrm{Proj}(k[X, Y]/(X^2))$ has dimension zero, so that the locally free sheaf defined by $(I/I^2) \otimes_R k$ is actually positive.

Similar examples can be computed from the other examples in section 3.

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