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Clearly  $v > 0$ ; moreover

$$u = \frac{\frac{p_{h-1}}{q_{h-1}} - \frac{q_h}{q_{h-1}} \frac{p_h}{q_h}}{1 - \frac{q_h}{q_{h-1}}} = \frac{p_{h-1} - p_h}{q_{h-1} - q_h} > 0.$$

It follows that this circle is entirely contained in the right half plane. A root  $x_j$  different from  $x_0$  (see Fig. 3) lies outside the circle (8.3) if  $h$  is large enough to have  $\Delta > |b - a|$ . It follows that  $\operatorname{Re} \mathcal{F}(x_j) < 0$ , and hence it is external to the circle (8.6) corresponding to the value  $h + 1$ . Hence the condition

$$F_{h-1} F_{h-2} \Delta > 1$$

ensures that the polynomial  $f_{h+1}$  is reduced.

## 8.2 THE CASE OF MULTIPLE ROOTS

Obreschkoff's Lemma 8.1 yields the following

**COROLLARY 8.3.** *Let  $f(x) = (x - x_1)(x - x_2) \cdots (x - x_r)$ , where  $x_i \in \mathbf{R}^+$ . Then*

$$f_1(x) = (x^2 + 2\rho x \cos \varphi + \rho^2) f(x), \quad \rho > 0, \quad |\varphi| < \frac{\pi}{r+2}$$

*has exactly  $r$  variations. More generally, a polynomial having  $r$  positive real roots and all its other roots in the sector*

$$S = \left\{ x \mid x = -\rho(\cos \varphi + i \sin \varphi), \quad \rho > 0, \quad |\varphi| < \frac{\pi}{r+2} \right\}$$

*has exactly  $r$  variations.*

This allows us to extend Vincent's theorem to the case of multiple roots. Suppose the polynomial  $f(x)$  has multiple roots, and let  $\Delta$  be their least distance. If  $h$  is sufficiently large to verify

$$F_h F_{h-1} \Delta > 1,$$

at most one root  $x_0$  lies in  $(a, b)$ , but since this root may have multiplicity  $r$ ,  $f_h$  has 0 or at least  $r$  variations. It will have exactly  $r$  variations if we can ensure that  $x_0 \in (a, b)$  and that the other transformed roots lie in the sector

$$S = \{y \mid \operatorname{Re} y < 0, \quad |\operatorname{Im} y| < |\tan \varphi| \cdot |\operatorname{Re} x|\},$$

where  $\varphi = \frac{\pi}{r+2}$ . Let  $s = \tan \frac{\pi}{r+2}$  and let us make the appropriate substitutions into (8.4). We have proved

THEOREM 8.4. *Let  $f(x)$  be a real polynomial of degree  $n$  whose roots are of multiplicity smaller than  $r$ . Let  $\gamma = [c_0, c_1, c_2, \dots]$  and, maintaining the previous notation, consider the polynomials*

$$f_{h+1} = (q_{h-1} + q_h x) f\left(\frac{p_{h-1} + p_h x}{q_{h-1} + q_h x}\right).$$

Let  $s = \tan \frac{\pi}{r+2}$ . If  $h$  satisfies

$$F_h F_{h-1} \Delta > \sqrt{1 + \frac{1}{s^2}} = \frac{1}{\sin \frac{\pi}{r+2}},$$

then the number of variations of  $f_{h+1}$  equals the multiplicity of the root in  $\left(\frac{p_{h-1}}{q_{h-1}}, \frac{p_h}{q_h}\right)$ .

REMARK 9. Obviously, letting  $s = \tan \frac{\pi}{n+2}$ , we can implement an algorithm to isolate the roots, without being forced to reduce the polynomial  $f(x)$  to one with simple roots.

REMARK 10. We conclude our paper by showing that our estimate of the size of  $h$  is asymptotically better than Chen's. Suppose we consider a polynomial whose roots are of multiplicity  $\leq r$  (which necessarily has degree  $n \geq r$ ). We have proved that the isolation of a root can be carried out in  $p$  steps, where  $p$  verifies

$$(8.7) \quad F_p F_{p-1} \Delta > \sqrt{1 + \frac{1}{\tan^2 \frac{\pi}{r+2}}} = \frac{1}{\sin \frac{\pi}{r+2}}.$$

We want to compare this integer with that needed by Chen's theorem, that is the smallest integer  $m = h + k$ , where  $h$  and  $k$  satisfy

$$(8.8) \quad F_h F_{h-1} \Delta > 1 \quad \text{and} \quad k > \frac{1}{2} \log_{\phi} r.$$

We know that

$$\sqrt{5} F_k \approx \phi^{k+1},$$

hence (8.7) becomes

$$\frac{\Delta}{5} \phi^{2p+1} > \frac{1}{\sin \frac{\pi}{r+2}}.$$

On the other hand, by (8.8) we have

$$(8.9) \quad \frac{\Delta}{5} \phi^{2h+1} > 1 \quad \text{and} \quad k > \frac{1}{2} \log_{\phi} r.$$

The second equality may be rewritten as

$$(8.10) \quad \phi^{2k} > r.$$

From the first inequality of (8.9) and (8.10) it follows that

$$\frac{\Delta}{5} \phi^{2h+1} \phi^{2k} = \frac{\Delta}{5} \phi^{2(h+k)+1} > r.$$

Hence

$$\frac{\Delta}{5} \phi^{2m+1} > r.$$

Since

$$r > \frac{1}{\sin \frac{\pi}{r+2}} \quad \text{for } r \geq 2,$$

$m \geq p$  for  $r$  sufficiently large and the proof is concluded.  $\square$

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