Zeitschrift: L'Enseignement Mathématique
Herausgeber: Commission Internationale de l'Enseignement Mathématique
Band: 44 (1998)
Heft: 3-4: L'ENSEIGNEMENT MATHÉMATIQUE

Artikel: NEW PROOF OF VINCENT'S THEOREM
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Kapitel: 5. USPENSKY'S PROOF OF VINCENT'S THEOREM
DOI: https://doi.org/10.5169/seals-63903

We find the new smaller interval $(\frac{2}{3}, 1)$. This remark explains the double role of Vincent's theorem, to isolate or to approximate the roots.

5. Uspensky's proof of Vincent's theorem

Uspensky had the great merit of rediscovering Vincent's theorem and of providing the first modern proof. He also tried to popularize the use of the theorem as a powerful tool to isolate the roots of algebraic equations, but there he was unsuccessful, and it was only at the end of the seventies, mainly by the work of Akritas, that the root separation algorithm acquired its present status.

To clarify the structure of the proof, which at first sight looks rather cumbersome, we extract part of its content as an independent lemma, which is of little interest in itself, but will be used also in the proof of Section 6.

LEMMA 5.1. *If the n positive numbers*

$$R_k = \binom{n-1}{k}(1+\delta_k)\,, \quad k = 0, 1, \ldots, n-1\,,$$

are such that $|\delta_k| < \dfrac{1}{n}$, then the $n-1$ inequalities

$$R_k^2 - R_{k-1}R_{k+1} > 0\,, \quad k = 1, \ldots, n-1 \tag{5.1}$$

hold.

Proof. The inequalities (5.1) may be written as

$$\frac{(1+\delta_k)^2}{(1+\delta_{k-1})(1+\delta_{k+1})} > 1 - \frac{n}{(n-k)(k+1)}\,. \tag{5.2}$$

If $\varepsilon = \max\{|\delta_k|\}$, the left hand side of (5.2) is greater than

$$\frac{(1-\varepsilon)^2}{(1+\varepsilon)^2} = 1 - \frac{4\varepsilon}{(1+\varepsilon)^2}\,.$$

Hence (5.2) holds if

$$\frac{4\varepsilon}{(1+\varepsilon)^2} < \frac{n}{(n-k)(k+1)}\,. \tag{5.3}$$

The minimum value of

$$\frac{n}{(n-k)(k+1)}$$

is

$$\frac{4n}{(n+1)^2} = \frac{4/n}{(1+\frac{1}{n})^2}.$$

It follows that (5.3) holds if $\varepsilon < \dfrac{1}{n}$. □

Now we give a precise statement, followed by a summary of the essential points of the proof [35, pp. 298–303].

THEOREM 5.2. *Let $f(x)$ be a real polynomial of degree n, without multiple roots, and with least roots distance Δ. Let $\gamma = [c_0, c_1, c_2, \dots]$, where the c_i are arbitrary positive integers for $i \geq 1$ and $c_0 \geq 0$, the k-th convergent being denoted by $\frac{p_k}{q_k}$. Let F_k denote the k-th term of the Fibonacci sequence (defined by $F_0 = F_1 = 1$, and $F_k = F_{k-1} + F_{k-2}$ for $k > 1$). If the integer h is such that*

$$F_{h-1}\frac{\Delta}{2} > 1 \quad \text{and} \quad \Delta F_h F_{h-1} > 1 + \frac{1}{\varepsilon_n},$$

where

$$\varepsilon_n = \Big(1 + \frac{1}{n}\Big)^{\frac{1}{n-1}} - 1,$$

then the polynomial given by (3.1),

$$f_{h+1}(x) = (q_{h-1} + q_h x)^n f\Big(\frac{p_{h-1} + p_h x}{q_{h-1} + q_h x}\Big),$$

has at most one variation [18]).

Proof. The first part of the proof partially follows Vincent's original argument. To simplify the notation, we set, as in Section 4, $a = \dfrac{p_{h-1}}{q_{h-1}}$, $b = \dfrac{p_h}{q_h}$, and we make the change of variable $x \leftarrow \dfrac{q_{h-1}}{q_h}x$. We are led to study the number of variations of the polynomial

$$\phi(x) = (1+x)^n f\Big(\frac{a+bx}{1+x}\Big),$$

the image of f under (4.1).

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[18]) In Uspensky's original proof [35] one reads $F_{h-1}\Delta > \frac{1}{2}$, probably a misprint that Uspensky had no time to correct, since he died before the publication of the book. The mistake, frequently reproduced, was corrected by Akritas in [3]. But our rereading of Uspensky's proof shows that this hypothesis is unnecessary.

Formulae (3.3) and (3.4) describe the behaviour of the linear and of the quadratic factors of $f(x)$. The hypothesis $F_{h-1}\frac{\Delta}{2} > 1$, which obviously implies the weaker hypothesis $F_h F_{h-1}\Delta > 1$, immediately allows us to prove that no complex root can be transformed into a root having a positive real part, and that at most one real root can be transformed into a positive real root.

Indeed, it follows from $F_h F_{h-1}\Delta > 1$ that

$$|b-a| = \frac{1}{q_h q_{h-1}} < \frac{1}{F_h F_{h-1}} < \Delta,$$

and consequently at most one real root lies in the interval (a,b). A quick look at formula (4.2) allows us to adapt the argument given in Remark 2 to the present situation, in order to exclude that a complex root lies in the circle having the real points a and b as the endpoints of a diameter.

Consider now the roots $x_0, x_1, \ldots, x_{n-1}$ of $f(x)$. If no root is in (a,b) then all the factors of the transformed polynomial $\phi(x)$ have positive coefficients, hence $\phi(x)$ has no variations, and the theorem is proved.

Let x_0 be the necessarily unique root of $f(x)$ lying in (a,b), and denote by x_j any other (real or complex) root.

The root x_j is transformed into

$$\xi_j = \frac{x_j-a}{b-x_j} = -1+\frac{b-a}{b-x_j} = -1+\alpha_j.$$

Now $|b-x_j| = |b-x_0+x_0-x_j| \geq |x_0-x_j| - |b-x_0| \geq \Delta - |b-a|$. It follows that

$$|\alpha_j| = \left|\frac{b-a}{b-x_j}\right| \leq \frac{|b-a|}{\Delta-|b-a|}.$$

Recalling that $|b-a| = \dfrac{1}{q_h q_{h-1}}$, and that $\Delta F_h F_{h-1} > 1+\dfrac{1}{\varepsilon_n}$, we conclude that

$$|\alpha_j| < \varepsilon_n.$$

The polynomial $\phi(x)$ is of the form

$$(x-\xi_0)(x+1+\alpha_1)(x+1+\alpha_2)\cdot\ldots\cdot(x+1+\alpha_{n-1}), \tag{5.4}$$

where $|\alpha_j| < \varepsilon_n$, for $j = 1,\ldots,n-1$. Let

$$(x+1+\alpha_1)(x+1+\alpha_2)\cdot\ldots\cdot(x+1+\alpha_{n-1}) = x^{n-1}+R_1x^{n-2}+\cdots+R_{n-2}x+R_{n-1}.$$

The coefficient R_k is given by the sum of $\binom{n-1}{k}$ products of the form $(1+\alpha_{i_1})(1+\alpha_{i_2})\cdot\ldots\cdot(1+\alpha_{i_k})$, and

$$|(1+\alpha_{i_1})(1+\alpha_{i_2})\cdot\ldots\cdot(1+\alpha_{i_k})-1| \leq (1+|\alpha_{i_1}|)\cdot\ldots\cdot(1+|\alpha_{i_k}|)-1$$
$$\leq (1+\varepsilon_n)^k - 1 \leq (1+\varepsilon_n)^{n-1} - 1 = \frac{1}{n}.$$

Hence

$$R_k = \binom{n-1}{k}(1+\delta_k),$$

with

$$|\delta_k| < \frac{1}{n}.$$

Now Lemma 5.1 may be applied to deduce that

$$R_{k+1}^2 - R_k R_{k-1} > 0,$$

and the argument used to conclude Vincent's proof also ensures that the transformed polynomial has only one variation. □

REMARK 7. In [3], Akritas observes that the last part of this proof is of enough interest to be stated as an independent Lemma:

If a real polynomial of degree $n > 1$ has one positive root, while all the other roots are concentrated in a circular neighbourhood of -1 with radius ε_n, then the polynomial has exactly one variation.

In [9], this Lemma is presented as a converse of the rule of signs. Another converse is given by a corollary to Obreschkoff's Lemma presented in Section 8. In any case, the problem is now reduced to that of evaluating an integer h such that the substitution (3.2) sends all the roots but the positive one into a neighbourhood of -1. Uspensky's proof, while ingenious, looks unnecessarily complicated, because the form (5.4) of the transformed polynomial does not reflect the fact that the complex roots of real polynomials appear in conjugate pairs. And instead of looking for a location of the roots ξ_k such that the number of variations does not increase, Uspensky, like Vincent, looks for a polynomial "close" to $(1+x)^{n-1}$. As a consequence he requires that the roots of the transformed polynomial lie in a very small neighbourhood of -1 (of radius ε_n, in fact), which in turn introduces the unnatural condition $F_h F_{h-1}\Delta > 1 + \dfrac{1}{\varepsilon_n}$. We shall prove that the result holds if $F_h F_{h-1}\Delta > \dfrac{2}{\sqrt{3}}$, and independently of n.