

Zeitschrift: L'Enseignement Mathématique
Herausgeber: Commission Internationale de l'Enseignement Mathématique
Band: 44 (1998)
Heft: 3-4: L'ENSEIGNEMENT MATHÉMATIQUE

Artikel: NEW PROOF OF VINCENT'S THEOREM
Autor: Alesina, Alberto / Galuzzi, Massimo
Kapitel: 3. A NECESSARY PRELIMINARY STEP : LAGRANGE
DOI: <https://doi.org/10.5169/seals-63903>

Nutzungsbedingungen

Die ETH-Bibliothek ist die Anbieterin der digitalisierten Zeitschriften auf E-Periodica. Sie besitzt keine Urheberrechte an den Zeitschriften und ist nicht verantwortlich für deren Inhalte. Die Rechte liegen in der Regel bei den Herausgebern beziehungsweise den externen Rechteinhabern. Das Veröffentlichen von Bildern in Print- und Online-Publikationen sowie auf Social Media-Kanälen oder Webseiten ist nur mit vorheriger Genehmigung der Rechteinhaber erlaubt. [Mehr erfahren](#)

Conditions d'utilisation

L'ETH Library est le fournisseur des revues numérisées. Elle ne détient aucun droit d'auteur sur les revues et n'est pas responsable de leur contenu. En règle générale, les droits sont détenus par les éditeurs ou les détenteurs de droits externes. La reproduction d'images dans des publications imprimées ou en ligne ainsi que sur des canaux de médias sociaux ou des sites web n'est autorisée qu'avec l'accord préalable des détenteurs des droits. [En savoir plus](#)

Terms of use

The ETH Library is the provider of the digitised journals. It does not own any copyrights to the journals and is not responsible for their content. The rights usually lie with the publishers or the external rights holders. Publishing images in print and online publications, as well as on social media channels or websites, is only permitted with the prior consent of the rights holders. [Find out more](#)

Download PDF: 15.01.2026

ETH-Bibliothek Zürich, E-Periodica, <https://www.e-periodica.ch>

3. A NECESSARY PRELIMINARY STEP: LAGRANGE

As Vincent repeatedly states, an important incentive to develop his own procedure for isolating the roots of an algebraic equation was given by Lagrange's *Traité de la résolution des équations numériques* [26], which collects and improves all the results in [23], [24], [25].

We begin by describing Lagrange's method for approximating a real root of an algebraic equation by a continued fraction expansion, in the oversimplified case of an algebraic equation which has a single positive root.

Actually Lagrange does much more than that, and via his famous *équation au carré des différences*, he gives a method which, in principle, amounts to a complete solution of the problem of approximating all the real roots. Nevertheless his solution is highly impractical and was strongly criticized by Fourier¹¹⁾.

Let x_0 be the unique positive root of a polynomial $f(x)$ of degree n , and let the simple continued fraction expansion¹²⁾ of x_0 be given by

$$x_0 = [c_0, c_1, c_2, \dots] = c_0 + \frac{1}{c_1 + \frac{1}{c_2 + \dots}}$$

where $c_0 \geq 0$ and $c_i > 0$ for $i > 0$. To avoid trivial cases, we suppose that $x_0 \notin \mathbf{Q}$.

Lagrange's method (see also [12]) consists in constructing a sequence of polynomials $\{f_h(x)\}$ defined recursively by

$$f_0(x) = f(x),$$

and, for $h \geq 0$,

$$f_{h+1}(x) = x^n f_h\left(c_h + \frac{1}{x}\right),$$

where c_h is the integer part (≥ 1 for $h \geq 1$) of the unique positive root

$$\alpha_h = \frac{1}{\alpha_{h-1} - c_{h-1}} \quad (\alpha_0 = x_0)$$

of the polynomial $f_h(x)$.

Denote the convergents of $x_0 = [c_0, c_1, c_2, \dots]$ by $\frac{p_0}{q_0}, \frac{p_1}{q_1}, \frac{p_2}{q_2}, \dots$. Then

¹¹⁾ [20, p.28]. We shall consider this quite interesting question in a subsequent paper.

¹²⁾ In this paper we make extensive use of the more familiar properties of continued fractions. A concise introduction to the subject is given in [12, Section 2].

(setting, as usual, $p_{-1} = 1$, $q_{-1} = 0$, $p_{-2} = 0$, $q_{-2} = 1$)

$$(3.1) \quad f_{h+1}(x) = (q_{h-1} + q_h x)^n f\left(\frac{p_{h-1} + p_h x}{q_{h-1} + q_h x}\right),$$

and equality (3.1) shows that¹³)

$$x_0 \in \left(\frac{p_{h-1}}{q_{h-1}}, \frac{p_h}{q_h}\right).$$

Each of the polynomials f_h has a unique positive root, and it will be proved later on that, for sufficiently large h , they each have a single variation in the sequence of their coefficients.

This apparently surprising result may be considered a particular case of Vincent's theorem which we are going to examine. But let us begin with a result of Lagrange.

A particularly favourable condition occurs when the variation is located between the coefficients of degree 1 and 0. The possibility of obtaining this particular situation was explored in [26, Note XII] for a general change of variables of the form

$$x \leftarrow \frac{p + rx}{q + sx}$$

and for a more general location of the roots, paving the way for future developments which led to Vincent's theorem.

The change of variables

$$x \leftarrow \frac{q}{s}x$$

does not affect the number of variations, consequently Lagrange limited himself to consider

$$(3.2) \quad x \leftarrow \frac{a + bx}{x + 1}.$$

THEOREM 3.1 (Lagrange). *Suppose that the real polynomial $f(x)$ of degree n has a single real root x_0 in the positive interval (a, b) [neither a or b being roots], and that no complex root has its real part in the same interval. If a is chosen sufficiently close to x_0 , then the polynomial*

$$\phi(x) = (1 + x)^n f\left(\frac{a + bx}{1 + x}\right)$$

has a unique variation, located between the coefficients of degree 0 and 1.

¹³) By (a, b) we denote the interval whose endpoints are a, b , but we do not suppose $a < b$. We also have $p_{i+1} = c_{i+1}p_i + p_{i-1}$ and $q_{i+1} = c_{i+1}q_i + q_{i-1}$.

Proof. Denote by x_1, x_2, \dots, x_{n-1} the other (real or complex) roots of $f(x)$. Consider first a real root x_j . According to (3.2), x_j is transformed into

$$(3.3) \quad \xi_j = \frac{x_j - a}{b - x_j},$$

which is positive if and only if $x_j \in (a, b)$, that is if and only if $x_j \equiv x_0$.

Hence the factor $x - x_0$ is transformed into the factor $x - \xi_0$, which has a sign variation, while every other linear factor $x - x_j$ ($j \neq 0$) is transformed into a factor of the form $x + p$, with $p \in \mathbf{R}^+$.

Consider now a complex root $x_k = \rho_k + i\sigma_k$. Under (3.2), x_k is carried into

$$(3.4) \quad \xi_k = \frac{\rho_k - a + i\sigma_k}{b - \rho_k - i\sigma_k} = \frac{(\rho_k - a)(b - \rho_k) - \sigma_k^2 + i(b - a)\sigma_k}{(b - \rho_k)^2 + \sigma_k^2}.$$

By hypothesis $\rho_k \notin (a, b)$, $(\rho_k - a)(b - \rho_k) < 0$, and hence

$$\operatorname{Re} \xi_k = \frac{(\rho_k - a)(b - \rho_k) - \sigma_k^2}{(b - \rho_k)^2 + \sigma_k^2} < 0.$$

Since complex roots appear in conjugate pairs, (3.2) transforms a quadratic factor of $f(x)$ of the form

$$(x - \rho - i\sigma)(x - \rho + i\sigma) = x^2 - 2\rho x + \rho^2 + \sigma^2$$

into a quadratic factor of the form

$$x^2 + 2Rx + R^2 + S^2,$$

where $R > 0$.

Therefore, $\phi(x)$ is of the form

$$K(x - \xi_0)(x + p) \cdot \dots \cdot (x^2 + 2Rx + R^2 + S^2) \cdot \dots,$$

where all the quantities $\xi_0, p, \dots, R, S, \dots$ are strictly positive, and

$$(3.5) \quad \xi_0 = \frac{x_0 - a}{b - x_0}.$$

Obviously the coefficients of the polynomial

$$(x + p) \cdot \dots \cdot (x^2 + 2Rx + R^2 + S^2) \cdot \dots$$

are strictly positive as well. Let us write this polynomial as

$$b_0 x^{n-1} + b_1 x^{n-2} + \dots + b_{n-2} x + b_{n-1},$$

where $b_i > 0$. Hence, up to the constant K ,

$$\begin{aligned}\phi(x) &= (x - \xi_0)(b_0x^{n-1} + b_1x^{n-2} + \cdots + b_{n-2}x + b_{n-1}) \\ &= b_0x^n + (b_1 - \xi_0b_0)x^{n-1} + (b_2 - \xi_0b_1)x^{n-2} + \cdots - \xi_0b_{n-1}.\end{aligned}$$

If in (3.5) a is so close to x_0 as to verify

$$\xi_0 < \min\left(\frac{b_1}{b_0}, \frac{b_2}{b_1}, \frac{b_3}{b_2}, \dots\right),$$

that is,

$$b_1 - \xi_0b_0 > 0, \quad b_2 - \xi_0b_1 > 0, \quad b_3 - \xi_0b_2 > 0, \quad \dots$$

then all the coefficients of $\phi(x)$, with the only exception of the constant term, are positive. \square

REMARK 2. The hypothesis on the real parts of the complex roots seems to be a bit artificial, like an 'ad hoc' expedient. A simpler hypothesis is that $|b - a|$ be less than the least distance Δ of all the roots, i.e., $|b - a| < \Delta$. The distance between two conjugate roots $\rho \pm i\sigma$ is 2σ , which entails $\Delta < 2\sigma$. The maximum value of the product $(\rho - a)(b - \rho)$, when $a \leq \rho \leq b$, is $\frac{1}{4}(b - a)^2$. It follows that

$$\frac{1}{4}(b - a)^2 < \frac{1}{4}\Delta^2 < \sigma^2,$$

and the real part of the transformed roots given by (3.4) is negative.

REMARK 3. The hypotheses Lagrange makes in *Note XII* are very stringent. By expanding the root into a continued fraction we can find a first integer h sufficiently large in order to have $\left|\frac{p_h}{q_h} - \frac{p_{h-1}}{q_{h-1}}\right| < \Delta$. This ensures that all the real parts of the roots transformed by

$$x \leftarrow \frac{p_{h-1} + p_hx}{q_{h-1} + q_hx}$$

are negative. Carrying on the process, we can find a second larger integer k such that $\left|\frac{p_k}{q_k} - \frac{p_{k-1}}{q_{k-1}}\right| < \varepsilon$. Choosing a between $\frac{p_k}{q_k}$ and $\frac{p_{k-1}}{q_{k-1}}$ and b between $\frac{p_h}{q_h}$ and $\frac{p_{h-1}}{q_{h-1}}$ we can satisfy Lagrange's condition. But isn't the knowledge of h and k equivalent to the possibility of approximating a root as closely as we desire? At first sight, *Note XII* appears pointless.