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3. A NECESSARY PRELIMINARY STEP: LAGRANGE

As Vincent repeatedly states, an important incentive to develop his own procedure for isolating the roots of an algebraic equation was given by Lagrange's *Traité de la résolution des équations numériques* [26], which collects and improves all the results in [23], [24], [25].

We begin by describing Lagrange's method for approximating a real root of an algebraic equation by a continued fraction expansion, in the oversimplified case of an algebraic equation which has a single positive root.

Actually Lagrange does much more than that, and via his famous *équation au carré des différences*, he gives a method which, in principle, amounts to a complete solution of the problem of approximating all the real roots. Nevertheless his solution is highly impractical and was strongly criticized by Fourier [11]).

Let x_0 be the unique positive root of a polynomial $f(x)$ of degree n, and let the simple continued fraction expansion [12]) of x_0 be given by

$$x_0 = [c_0, c_1, c_2, \ldots] = c_0 + \cfrac{1}{c_1 + \cfrac{1}{c_2 + \ddots}}$$

where $c_0 \geq 0$ and $c_i > 0$ for $i > 0$. To avoid trivial cases, we suppose that $x_0 \notin \mathbf{Q}$.

Lagrange's method (see also [12]) consists in constructing a sequence of polynomials $\{f_h(x)\}$ defined recursively by

$$f_0(x) = f(x),$$

and, for $h \geq 0$,

$$f_{h+1}(x) = x^n f_h\Big(c_h + \frac{1}{x}\Big),$$

where c_h is the integer part (≥ 1 for $h \geq 1$) of the unique positive root

$$\alpha_h = \frac{1}{\alpha_{h-1} - c_{h-1}} \qquad (\alpha_0 = x_0)$$

of the polynomial $f_h(x)$.

Denote the convergents of $x_0 = [c_0, c_1, c_2, \ldots]$ by $\dfrac{p_o}{q_0}, \dfrac{p_1}{q_1}, \dfrac{p_2}{q_2}, \ldots$. Then

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[11]) [20, p. 28]. We shall consider this quite interesting question in a subsequent paper.

[12]) In this paper we make extensive use of the more familiar properties of continued fractions. A concise introduction to the subject is given in [12, Section 2].

(setting, as usual, $p_{-1}=1$, $q_{-1}=0$, $p_{-2}=0$, $q_{-2}=1$)

$$f_{h+1}(x)=(q_{h-1}+q_h x)^n f\Big(\frac{p_{h-1}+p_h x}{q_{h-1}+q_h x}\Big)\,, \tag{3.1}$$

and equality (3.1) shows that [13])

$$x_0 \in \Big(\frac{p_{h-1}}{q_{h-1}},\frac{p_h}{q_h}\Big)\,.$$

Each of the polynomials f_h has a unique positive root, and it will be proved later on that, for sufficiently large h, they each have a single variation in the sequence of their coefficients.

This apparently surprising result may be considered a particular case of Vincent's theorem which we are going to examine. But let us begin with a result of Lagrange.

A particularly favourable condition occurs when the variation is located between the coefficients of degree 1 and 0. The possibility of obtaining this particular situation was explored in [26, *Note XII*] for a general change of variables of the form

$$x \leftarrow \frac{p+rx}{q+sx}$$

and for a more general location of the roots, paving the way for future developments which led to Vincent's theorem.

The change of variables

$$x \leftarrow \frac{q}{s}x$$

does not affect the number of variations, consequently Lagrange limited himself to consider

$$x \leftarrow \frac{a+bx}{x+1}\,. \tag{3.2}$$

THEOREM 3.1 (Lagrange). *Suppose that the real polynomial $f(x)$ of degree n has a single real root x_0 in the positive interval (a,b) [neither a or b being roots], and that no complex root has its real part in the same interval. If a is chosen sufficiently close to x_0, then the polynomial*

$$\phi(x)=(1+x)^n f\Big(\frac{a+bx}{1+x}\Big)$$

has a unique variation, located between the coefficients of degree 0 and 1.

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[13]) By (a,b) we denote the interval whose endpoints are a,b, but we do not suppose $a<b$. We also have $p_{i+1}=c_{i+1}p_i+p_{i-1}$ and $q_{i+1}=c_{i+1}q_i+q_{i-1}$.

Proof. Denote by $x_1, x_2, \ldots, x_{n-1}$ the other (real or complex) roots of $f(x)$. Consider first a real root x_j. According to (3.2), x_j is transformed into

$$\xi_j = \frac{x_j - a}{b - x_j}, \tag{3.3}$$

which is positive if and only if $x_j \in (a, b)$, that is if and only if $x_j \equiv x_0$.

Hence the factor $x - x_0$ is transformed into the factor $x - \xi_0$, which has a sign variation, while every other linear factor $x - x_j$ $(j \neq 0)$ is transformed into a factor of the form $x + p$, with $p \in \mathbf{R}^+$.

Consider now a complex root $x_k = \rho_k + i\sigma_k$. Under (3.2), x_k is carried into

$$\xi_k = \frac{\rho_k - a + i\sigma_k}{b - \rho_k - i\sigma_k} = \frac{(\rho_k - a)(b - \rho_k) - \sigma_k^2 + i(b-a)\sigma_k}{(b - \rho_k)^2 + \sigma_k^2}. \tag{3.4}$$

By hypothesis $\rho_k \notin (a, b)$, $(\rho_k - a)(b - \rho_k) < 0$, and hence

$$\operatorname{Re} \xi_k = \frac{(\rho_k - a)(b - \rho_k) - \sigma_k^2}{(b - \rho_k)^2 + \sigma_k^2} < 0.$$

Since complex roots appear in conjugate pairs, (3.2) transforms a quadratic factor of $f(x)$ of the form

$$(x - \rho - i\sigma)(x - \rho + i\sigma) = x^2 - 2\rho x + \rho^2 + \sigma^2$$

into a quadratic factor of the form

$$x^2 + 2Rx + R^2 + S^2,$$

where $R > 0$.

Therefore, $\phi(x)$ is of the form

$$K(x - \xi_0)(x + p) \cdot \ldots \cdot (x^2 + 2Rx + R^2 + S^2) \cdot \ldots;$$

where all the quantities $\xi_0, p, \ldots, R, S, \ldots$ are strictly positive, and

$$\xi_0 = \frac{x_0 - a}{b - x_0}. \tag{3.5}$$

Obviously the coefficients of the polynomial

$$(x + p) \cdot \ldots \cdot (x^2 + 2Rx + R^2 + S^2) \cdot \ldots$$

are strictly positive as well. Let us write this polynomial as

$$b_0 x^{n-1} + b_1 x^{n-2} + \cdots + b_{n-2} x + b_{n-1},$$

where $b_i > 0$. Hence, up to the constant K,

$$\begin{aligned}\phi(x) &= (x-\xi_0)(b_0x^{n-1} + b_1x^{n-2} + \cdots + b_{n-2}x + b_{n-1}) \\ &= b_0x^n + (b_1 - \xi_0 b_0)x^{n-1} + (b_2 - \xi_0 b_1)x^{n-2} + \cdots - \xi_0 b_{n-1}\,.\end{aligned}$$

If in (3.5) a is so close to x_0 as to verify

$$\xi_0 < \min\Big(\frac{b_1}{b_0}, \frac{b_2}{b_1}, \frac{b_3}{b_2}, \ldots\Big)\,,$$

that is,

$$b_1 - \xi_0 b_0 > 0, \quad b_2 - \xi_0 b_1 > 0, \quad b_3 - \xi_0 b_2 > 0, \ \ldots$$

then all the coefficients of $\phi(x)$, with the only exception of the constant term, are positive. $\square$

REMARK 2. The hypothesis on the real parts of the complex roots seems to be a bit artificial, like an 'ad hoc' expedient. A simpler hypothesis is that $|b-a|$ be less than the least distance Δ of all the roots, i.e., $|b-a| < \Delta$. The distance between two conjugate roots $\rho \pm i\sigma$ is 2σ, which entails $\Delta < 2\sigma$. The maximum value of the product $(\rho - a)(b-\rho)$, when $a \le \rho \le b$, is $\frac{1}{4}(b-a)^2$. It follows that

$$\tfrac{1}{4}(b-a)^2 < \tfrac{1}{4}\Delta^2 < \sigma^2\,,$$

and the real part of the transformed roots given by (3.4) is negative.

REMARK 3. The hypotheses Lagrange makes in *Note XII* are very stringent. By expanding the root into a continued fraction we can find a first integer h sufficiently large in order to have $\left|\dfrac{p_h}{q_h} - \dfrac{p_{h-1}}{q_{h-1}}\right| < \Delta$. This ensures that all the real parts of the roots transformed by

$$x \leftarrow \frac{p_{h-1} + p_h x}{q_{h-1} + q_h x}$$

are negative. Carrying on the process, we can find a second larger integer k such that $\left|\dfrac{p_k}{q_k} - \dfrac{p_{k-1}}{q_{k-1}}\right| < \varepsilon$. Choosing a between $\dfrac{p_k}{q_k}$ and $\dfrac{p_{k-1}}{q_{k-1}}$ and b between $\dfrac{p_h}{q_h}$ and $\dfrac{p_{h-1}}{q_{h-1}}$ we can satisfy Lagrange's condition. But isn't the knowledge of h and k equivalent to the possibility of approximating a root as closely as we desire? At first sight, *Note XII* appears pointless.