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$$\frac{1}{\epsilon} \sum_i \text{Int}(f_i) > \sum_i \#(f_i) = \#(\Delta) + \frac{1}{2} \sum_i \text{Int}(f_i),$$

whence $\frac{2-\epsilon}{2\epsilon} \sum_i \text{Int}(f_i) > \#(\Delta)$ or $\frac{2-\epsilon}{\epsilon} I(\Delta) > \#(\Delta)$. Since $\epsilon = 2\theta/(1 + \theta)$, we obtain $I(\Delta) > \theta\#(\Delta)$, which contradicts the θ -condition.

In fact, if the reduced diagram Δ is not simple, it is a union of simple diagrams linked by bridges. So each of its parts, which is a simple diagram, defines another reduced diagram (relative to another word), so the inequality holds for every part of Δ which is a simple diagram. We conclude by saying that increasing the number of external edges does not affect the inequality. \square

Proof of 4.3. By Lemma 4.2, it is sufficient to prove that the (ϵ, x_0) -balanced and θ -conditions imply that every non trivial reduced word in \mathbf{F}_X which vanishes in Γ contains at least one $x_0^{\pm 1}$.

Let us choose such a word ω and Δ a reduced diagram of ω . By Lemma 4.4, there exists a cell f with border equal to one $r \in R^*$, such that

$$\text{Int}(f) \leq \frac{2\theta}{1 + \theta} \#(f) = \frac{2\theta}{1 + \theta} |r| < \epsilon |r| \leq n_{x_0}(r),$$

because $\theta < \epsilon/(2 - \epsilon)$. As there are more occurrences of x_0 or x_0^{-1} than the number of internal edges, it means that some occurrences of x_0 or x_0^{-1} will be external edges, i.e. will be in the border of Δ which is ω . \square

We are now able to prove the main theorem.

Proof of theorem 1.1. By Proposition 4.3, for a finite presentation $\langle X | R \rangle$, we know that being (ϵ, x_0) -balanced and satisfying a θ -condition is sufficient to ensure that $X - \{x_0\}$ freely generates a free subgroup in Γ . But by Corollary 3.2 and [13, Theorem 2], these two conditions are generic and so is the conjunction of these two conditions. \square

5. SPECTRAL ESTIMATES FOR ADJACENCY OPERATORS ON CAYLEY GRAPHS

The existence of a free subgroup generated by $X - \{x_0\}$ gives an upper bound for the spectral value of the adjacency operator on the Cayley graph of $\Gamma = \langle X | R \rangle$ associated with the symmetric generating system $S = X \cup X^{-1}$.

We briefly recall some definitions and notations. The Cayley graph $G(\Gamma, X)$ of Γ associated with S has its set of vertices in bijection with Γ and two

vertices g_1 and g_2 are linked by an edge if and only $g_1^{-1}g_2 \in S$. A graph is completely determined by its adjacency operator and, in the case of Cayley graphs, the adjacency operator h_S can be expressed in terms of the right regular representation ρ acting on $l^2(\Gamma)$ as

$$h_S = \frac{1}{\#S} \sum_{s \in S} \rho(s).$$

The spectral properties of h_S capture some information about the pair $(\Gamma, S = X \cup X^{-1})$. For example, Kesten proved

THEOREM (Kesten [10], [9]). *Let Γ be a finitely generated group, let X be a finite generating system and set $S = X \cup X^{-1}$.*

a) The following are equivalent:

i) $\|h_S\| = 1$;

ii) Γ is amenable.

b) Assume that $\#X \geq 2$; then $\frac{\sqrt{2(\#X)-1}}{\#X} \leq \|h_S\|$. Equality holds if and only if Γ is isomorphic to the free group \mathbf{F}_X generated by X .

This enables us to give an easy proposition which was pointed out to us by Pierre de la Harpe.

PROPOSITION 5.1. *Let $\Gamma = \langle X | R \rangle$ be a finite presentation of a group Γ with $\#X \geq 2$. If $X \cap X^{-1} = \emptyset$ and if there exists $x_0 \in X$ such that $X - \{x_0\}$ generates a free subgroup in Γ then*

$$\frac{\sqrt{2(\#X)-1}}{\#X} \leq \|h_S\| \leq \frac{\sqrt{2(\#X)-3} + 1}{\#X}.$$

Proof. The first inequality is just Kesten's. To prove the second one, set $X' = X - \{x_0\}$, $S' = X' \cup (X')^{-1}$. Then we can write

$$(\#S)h_S = \rho(x_0) + \rho(x_0)^{-1} + \sum_{s \in S'} \rho(s).$$

As X' freely generates a free group, by Kesten's result, we obtain that

$$\left\| \sum_{s \in S'} \rho(s) \right\| = 2\sqrt{2(\#X')-1} = 2\sqrt{2(\#X)-3}.$$

So

$$\|(\#S)h_S\| \leq 2 + \left\| \sum_{s \in S'} \rho(s) \right\| = 2 + 2\sqrt{2(\#X)-3}.$$

And as $X \cap X^{-1} = \emptyset$, $\#S = 2\#X$ and we get

$$\|h_S\| \leq \frac{1 + \sqrt{2(\#X) - 3}}{\#X}.$$

□

The last proposition and Theorem 1.1 permit us to give generic upper bounds for $\|h_S\|$. Note that this upper bound is non trivial only for $\#X \geq 3$.

COROLLARY 5.2. *For a presentation $\Gamma = \langle X | R \rangle$ with $\#X = k \geq 3$ and $\#R = m$ fixed, the inequalities $\frac{\sqrt{2(\#X)-1}}{\#X} \leq \|h_S\| \leq \frac{\sqrt{2(\#X)-3}+1}{\#X}$ are generically true.*

Proof. Let $x, y \in X$. Since $k \geq 3$, there exists $x_0 \in X$ distinct from x and y . By Theorem 1.1, the subgroup generated by $X - \{x_0\}$ is generically free; in particular $xy \neq e$ in Γ . This shows that, generically, $X \cap X^{-1} = \emptyset$. The corollary follows then by combining Theorem 1.1 with Proposition 5.1. □

It was proved by Grigorchuk [7, Theorem 7.1] that for any fixed $\epsilon > 0$, any group satisfying the small cancellation hypothesis $C'(1/6)$ and such that the length of every relation is sufficiently large satisfies

$$\frac{\sqrt{2(\#X) - 1}}{\#X} < \|h_S\| < \frac{\sqrt{2(\#X) - 1}}{\#X} + \epsilon.$$

This corresponds to the intuitive idea that when the relations are long and do not cancel too much, the Cayley graph looks like a tree in some ball of large radius.

Champetier (in [4]) generalised this theorem, by replacing the small cancellation $C'(1/6)$, by a weaker condition defined by:

DEFINITION (Champetier). A finite presentation $\langle x_1, \dots, x_k | r_1, \dots, r_m \rangle$ satisfies the $A(C)$ condition for $C > 0$, if for every word ω in \mathbf{F}_X representing the identity in $\langle X | R \rangle$, there exists a diagram Δ representing ω such that, if there are l_i 2-cells contained in Δ having the relation r_i as border, then

$$\sum_{i=1}^m l_i |r_i| \leq C|\omega|.$$

With that definition, the precise statement of Champetier's theorem is:

THEOREM (Champetier). *Let C be a positive constant. For every $\epsilon > 0$, there exists an integer n_0 such that for every presentation $\Gamma = \langle X | R \rangle$, with $\#R = m$, satisfying $A(C)$ and $n_0 \leq \inf\{|r| \mid r \in R\}$, the following inequalities hold:*

$$\frac{\sqrt{2(\#X) - 1}}{\#X} \leq \|h_S\| \leq \left(1 + \frac{\epsilon}{2}\right) \frac{\sqrt{2(\#X) - 1}}{\#X}.$$

Assume the presentation satisfies a θ -condition: then $I(\Delta) < \theta\#(\Delta) = \theta(I(\Delta) + |\omega|)$, for any reduced diagram associated with ω . As

$$\sum_{i=1}^m l_i |r_i| = \sum_{2\text{-cell } f \subset \Delta} \#(f) \leq 2I(\Delta) + E(\Delta),$$

it is easy to see that the θ -condition implies $A(\frac{2\theta}{1-\theta} + 1)$. So Champetier's theorem and the genericity of the θ -condition imply:

COROLLARY 5.3. *For every $\epsilon > 0$, every fixed $\#X = k$ and every fixed $\#R = m$, $\|h_S\|$ is generically close to $\frac{\sqrt{2(\#X)-1}}{\#X}$.*

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