

**Zeitschrift:** L'Enseignement Mathématique  
**Herausgeber:** Commission Internationale de l'Enseignement Mathématique  
**Band:** 44 (1998)  
**Heft:** 3-4: L'ENSEIGNEMENT MATHÉMATIQUE

**Artikel:** THE REPRESENTATION THEORY OF AFFINE TEMPERLEY-LIEB ALGEBRAS  
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**Kapitel:** §5. DECOMPOSITION MATRICES  
**DOI:** <https://doi.org/10.5169/seals-63902>

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## (4.8) COROLLARY.

(1) If  $n$  is an odd positive integer, then Jones' annular algebra  $\mathbf{J}(n)$  (with parameter  $\delta = -q - q^{-1}$ ) is non-semisimple if and only if there exist distinct odd integers  $s, t \in \mathbf{n}$  such that  $q^{st} = 1$ .

(2) If  $n$  is an even positive integer, then Jones' annular algebra  $\mathbf{J}(n)$  (with parameter  $\delta = -q - q^{-1}$ ) is non-semisimple if and only if  $q^{\frac{n}{2}+1} = 1$  or there exist distinct even integers  $s, t \in \mathbf{n}$  such that  $q^{\frac{st}{2}} = 1$ .

*Proof.* By [GL, 3.8] the algebra is semisimple precisely when the bilinear pairing  $\langle \cdot, \cdot \rangle_{t,z}$  is non-degenerate on each cell representation (of  $\mathbf{J}(n)$ ); this condition is equivalent to the vanishing of the determinant  $\det G_{t,z}(n)$ , which by (4.7) immediately yields the stated condition.  $\square$

## §5. DECOMPOSITION MATRICES

(5.1) THEOREM. Let  $R$  be an algebraically closed field of characteristic zero and  $q$  a nonzero element of  $R$ . Let  $\preceq$  be the weakest partial order on the set  $\Lambda^a$  defined in (2.6) such that  $(t, z) \preceq (s, y)$  if  $(t, z)$  and  $(s, y)$  satisfy the hypotheses of Theorem (3.4) for  $q$  or  $q^{-1}$ . If  $(t, z) \in \Lambda^a$ ,  $n \in \mathbf{Z}_{\geq 0}$  and  $(s, y) \in \Lambda^a(n)$ , then the multiplicity of the irreducible  $\mathbf{T}^a(n)$ -module  $L_{s,y}(n)$  in the cell representation  $W_{t,z}(n)$  of (2.6) is one if  $(s, y) \succeq (t, z)$  and zero otherwise.

*Proof.* Let  $R$  be a field and  $q \in R$ . Let  $p: R[y] \rightarrow R$  be the  $R$ -algebra homomorphism defined by  $y \mapsto q + q^{-1}$ , where  $y$  is an indeterminate over  $R$ . Suppose  $W$  is a free  $R[y]$ -module of finite rank with an  $R[y]$ -bilinear form  $\langle \cdot, \cdot \rangle: W \times W \rightarrow R[y]$ . If  $R$  is regarded as a  $R[y]$ -module via the homomorphism  $p$ , the free  $R$ -module  $W_R = R \otimes_{R[y]} W$  inherits an  $R$ -bilinear form  $\langle \cdot, \cdot \rangle_R: W_R \times W_R \rightarrow R$  given by  $\langle 1 \otimes x, 1 \otimes y \rangle_R = p(\langle x, y \rangle)$ . Choose  $R[y]$ -bases  $B_1$  and  $B_2$  of  $W$  and let  $G$  denote the associated gram matrix of  $\langle \cdot, \cdot \rangle$ . If this form is nonsingular (i.e.  $\det G \neq 0$ ), then it may be shown that the multiplicity of the polynomial  $y - q - q^{-1}$  in the determinant  $\det G$  is greater than or equal to the  $R$ -dimension of the radical of  $\langle \cdot, \cdot \rangle_R$ . In fact if we denote the multiplicity of the polynomial  $y - q - q^{-1}$  in  $f \in R[y]$  by  $\text{mult}(f)$ , then

$$\text{mult}(\det G) = \sum_{i \geq 0} \dim \text{rad}^i$$

where  $\text{rad}^i$  denotes the image under  $\phi: W \rightarrow W_R: w \mapsto 1 \otimes w$  of the  $R[y]$ -submodule  $\{w \in W \mid \langle w, v \rangle \in (y - q - q^{-1})^i R[y] \text{ for any } v \in W\}$ .

(Since  $R[y]$  is a principal ideal domain, row and column operations may be used to reduce the proof of this fact to the easy case when  $G$  is diagonal.) We shall use this elementary result to give a bound for the dimension of the radical of the restriction of  $\langle \cdot, \cdot \rangle_{t,z}$  to  $W_{t,z}^s(n)$ .

Let  $t \leq s$  be non-negative integers of the same parity,  $n \in \mathbf{Z}_{\geq 0}$  and assume the hypotheses of the statement. Consider  $\mathbf{T}_{(R[x], -x)}^a$ . We shall compute the determinant of the gram matrix  $G_{t,0}^s(n)$  as a polynomial in  $y = x + x^{-1}$ . Our first goal is to compute the multiplicity of  $y - q - q^{-1}$  in this polynomial, i.e. to compute  $\text{mult}(\det G_{t,0}^s(n))$ . Let  $l$  denote the order of  $q^2$ . Since  $[n]_x$  and  $\begin{bmatrix} n \\ i \end{bmatrix}_x$  are polynomials in  $y = x + x^{-1}$  we may speak of the multiplicity of  $y - q - q^{-1}$  in these polynomials and it is straightforward that

$$\text{mult}[n]_x = \begin{cases} 1 & \text{if } l \neq 1, \infty \text{ and } l \text{ divides } n, \\ 0 & \text{otherwise,} \end{cases}$$

$$\text{and hence } \text{mult} \begin{bmatrix} n \\ i \end{bmatrix}_x = \begin{cases} 1 & \text{if } l \neq \infty \text{ and } \text{res}_l(n) < \text{res}_l(i), \\ 0 & \text{otherwise,} \end{cases}$$

where  $\text{res}_l(n) \in \{0, 1, \dots, l-1\}$  is determined by  $\text{res}_l(n) \equiv n \pmod{l}$ .

We next give an expression for  $\text{mult}([t; r]_x / [s; r]_x)$ . Let  $r \geq s$  have the same parity as  $s$  (or  $t$ ) and write  $X = \{0, 1, \dots, l-1\}$ . Then there exist unique elements  $k \in \mathbf{Z}$  and  $\bar{r} \in X$  such that  $r = kl + \bar{r}$ . Let  $\bar{t}$  denote the unique element of  $X$  such that  $kl + \bar{t} \equiv \pm t \pmod{2l}$ ; define  $\bar{s}$  similarly. Define:

$$\epsilon_t^s(r) = \begin{cases} 1 & \text{if } \bar{s} \leq \bar{r} < \bar{t}, \\ -1 & \text{if } \bar{t} \leq \bar{r} < \bar{s}, \\ 0 & \text{otherwise.} \end{cases}$$

The function  $\epsilon_t^s(r)$  satisfies

- (1)  $\epsilon_t^s(r) = \epsilon_s^{-t}(r) = \epsilon_s^{t+2l}(r)$
- (2)  $\epsilon_t^s(r) = -\epsilon_s^t(r)$ .

It is easy to see that if  $0 \leq t \leq s \leq r$ , then

$$\epsilon_t^s(r) = \text{mult}([t; r]_x / [s; r]_x).$$

By Corollary (4.5) and Proposition (4.6), we have

$$(5.1.1) \quad \text{mult}(\det G_{t,0}^s(n)) = \sum_{\substack{r \geq s \\ r \equiv t \pmod{2}}} \epsilon_t^s(r) \dim W_r(n).$$

If  $l = \infty$  or  $s \equiv t$  or  $-t \pmod{2l}$ , then  $\epsilon_t^s(r) = 0$  and so the multiplicity (5.1.1) is zero. For the remainder of this paragraph, assume that  $l \neq \infty$  and

$s \not\equiv \pm t \pmod{2l}$ . Let  $t' \in \mathbf{Z}$  be minimal such that  $t' > s$  and  $t' \pm t \equiv 0 \pmod{2l}$ . Let  $s' \in \mathbf{Z}$  be maximal such that  $t' > s'$  and  $s' \pm s \equiv 0 \pmod{2l}$ . Then  $s + 2l > t' > s' \geq s > t$ . Now in order to compute  $\text{mult}(\det G_{t,0}^s(n))$ , we partition the sum on the right side of (5.1.1) into three parts:

- (1)  $s \leq r < s'$ .
- (2)  $s' \leq r < t'$ .
- (3)  $t' \leq r$ .

For the terms in the first part,  $\epsilon_t^s(r) = 0$ . For those in the second part  $\epsilon_t^s(r) = 1$  and consequently, these terms contribute  $\dim W_{s',0}^{t'}(n) = \sum_{s' \leq r < t'} \dim W_r(n)$  to the sum. The terms in the third part have  $\epsilon_t^s(r) = -\epsilon_{s'}^{t'}(r)$  (by properties (1) and (2) of the function  $\epsilon_t^s(r)$ ) and so these terms contribute  $\text{mult}(\det G_{s',0}^{t'}(n))$  to the sum.

It follows that

$$(5.1.2) \quad \text{mult}(\det G_{t,0}^s(n)) = \dim W_{s',0}^{t'}(n) - \text{mult}(\det G_{s',0}^{t'}(n)).$$

Note that equation (5.1.2) should be interpreted as a recurrence relation for  $\text{mult}(\det G_{t,0}^s(n))$ , which together with the initial condition  $\text{mult}(\det G_{t,0}^s(n)) = 0$  if  $n \leq t$ , determines the multiplicity.

Now fix  $n \in \mathbf{Z}_{\geq 0}$ . Choose  $(t, z) \in \Lambda^a$  such that  $t \leq n$  and  $t \equiv n \pmod{2}$ . To prove the Theorem, we shall construct a composition series for  $W_{t,z}(n)$ .

If  $(t, z)$  is maximal in  $\Lambda^a(n)$  (with respect to  $\prec$ ), then it follows from Corollary 4.4 and Proposition 4.6, that  $\text{rad}_{t,z}(n) = 0$ ; hence the irreducible module  $L_{t,z}(n)$  coincides with  $W_{t,z}(n)$  and the statement follows.

Assume that  $(t, z)$  is not a maximal element of  $\Lambda^a(n)$  and choose  $(s, y) \in \Lambda^a(n)$  such that  $(s, y) \succ (t, z)$  and  $s$  is minimal with respect to this property. Then the hypotheses of Theorem (3.4) are satisfied (possibly after replacing  $q$  by  $q^{-1}$ ) and so we have an injective homomorphism  $\theta_n: W_{s,y}(n) \rightarrow W_{t,z}(n)$  of  $\mathbf{T}_{R,q}^a(n)$ -modules. The quotient  $Q = W_{t,z}(n)/\text{Im } \theta_n$  has basis  $\mu + \text{Im } \theta_n$  indexed by standard diagrams  $\mu: t \rightarrow n$  of rank strictly less than  $(s-t)/2$ . By (2.8), the image of  $\theta_n$  is contained in  $\text{rad}_{t,z}(n)$ , whence the bilinear form  $\langle \ , \ \rangle_{t,z}$  descends to  $Q \times Q \rightarrow R$ ; its gram matrix (with respect to the basis above) is  $G_{t,z}^s(n)$  and  $L_{t,z}(n)$  is the quotient of  $Q$  by its radical which we denote by  $\text{rad}_{t,z}^s(n)$ . Consider, for the moment,  $\mathbf{T}_{R[x],x}^a$ . The multiplicity  $\text{mult}(\det G_{t,z}^s(n)) = \text{mult}(\det G_{t,0}^s(n))$  by Corollary (4.4); it follows from the remarks concerning linear algebra at the beginning of this proof that

$$(5.1.3) \quad \dim \text{rad}_{t,z}^s(n) \leq \text{mult}(\det G_{t,0}^s(n)).$$

If the order  $l$  (of  $q^2$ ) is infinite, then  $(s, y)$  is the unique element of  $\Lambda^a$  such that  $(s, y) \succ (t, z)$ . If  $l$  is finite and  $s \equiv t$  or  $-t \pmod{2l}$ , then  $(s, y)$  is the unique element of  $\Lambda^a$  which covers  $(t, z)$ . In either case, we saw above that  $\text{mult}(\det G_{t,0}^s(n)) = 0$  and so  $\text{rad}_{t,z}^s(n) = 0$ . Therefore  $Q = L_{t,z}(n)$  and the composition factors of  $W_{t,z}(n)$  are  $L_{t,z}(n)$  together with those of  $W_{s,y}(n)$ , as required.

Assume that  $l$  is finite and  $s \not\equiv \pm t \pmod{2l}$ . Let  $s'$  and  $t'$  be as above and  $y' = \epsilon y^{-1}$  where  $\epsilon = q^{(s+s')/2} = \pm 1$ . Then  $(s', y')$  is the unique element of  $\Lambda^a$  such that  $(s', y') \succ (t, z)$  and  $(s', y') \not\preceq (s, y)$ . If  $s' > n$ , then the initial condition associated with (5.1.2) shows that  $\text{mult}(\det G_{t,0}^s(n)) = 0$  and so  $\text{rad}_{t,z}^s(n) = 0$ ; hence  $Q = L_{t,z}(n)$  and the statement of (5.1) follows as in the previous paragraph.

Finally, assume that  $s' \leq n$ . By Theorem (3.4) (with  $q^{-1}$  replacing  $q$ ), there exists an injective  $\mathbf{T}^a(n)$ -homomorphism  $\theta'_n: W_{s',y'}(n) \rightarrow W_{t,z}(n)$ . Thus  $L_{s',y'}(n)$  is a composition factor of  $W_{t,z}(n)$ . Arguing by induction in the poset  $\Lambda^a$ , we may assume that  $L_{s',y'}(n)$  is not a composition factor of  $W_{s,y}(n) \cong \text{Im}(\theta_n)$  since  $(s', y') \not\preceq (s, y)$ . It follows that the irreducible module  $L_{s',y'}(n)$  is a composition factor of  $\text{rad}_{t,z}^s(n)$  and we have, using (5.1.3),

$$\dim L_{s',y'}(n) \leq \dim \text{rad}_{t,z}^s(n) \leq \text{mult}(\det G_{t,0}^s(n)).$$

Arguing as above with  $(s', y')$  in place of  $(t, z)$  we have

$$\dim L_{s',y'}(n) = \dim Q' - \dim(\text{rad}_{s',y'}^{t'}(n)) \geq \dim W_{s',y'}^{t'}(n) - \text{mult}(\det G_{t',0}^{s'}(n)).$$

Now (5.1.2) asserts that the two ends of this chain of inequalities are equal. Hence we have equality at every step and in particular  $L_{s',y'}(n)$  is isomorphic to  $\text{rad}_{t,z}^s(n)$ . Thus the composition factors of  $W_{t,z}(n)$  are  $L_{t,z}(n)$  (if  $q^2 \neq 0$  or  $(t, z) \neq (0, q)$ ) and  $L_{s',y'}(n)$  together with those of  $W_{s,y}(n)$ , as required.  $\square$

(5.2) COROLLARY. Assume the hypotheses and notation of Theorem 5.1 and let  $\mathbf{J}(n)$  be Jones' annular algebra (see (2.10)). If  $(t, z) \in \Lambda^a(n)$  is such that  $t > 0$  and  $z^t = 1$ , then the  $\mathbf{J}(n)$ -module  $W_{t,z}(n)$  has composition factors  $L_{s,y}(n)$  indexed by  $(s, y) \in \Lambda^a(n)$  such that  $(s, y) \succeq (t, z)$ . The remaining cell module  $W_{0,q}/M$  (2.10) has composition factors  $L_{s,y}(n)$  indexed by  $(s, y) \in \Lambda^a(n)$  such that  $(s, y) \succeq (0, q)$  and  $(s, y) \not\preceq (2, 1)$ .

The next result is implicit in [DJ] and may be found in [Ma], [GW] and [W].

(5.3) THEOREM. Let  $R$  be a field of characteristic zero, let  $q$  be a nonzero element of  $R$  and let  $\mathbf{T}(n) = \mathbf{T}_{R,q}(n)$  be the Temperley-Lieb algebra over  $R$ , with parameter  $q$ . If  $n, t \in \mathbf{Z}_{\geq 0}$  and  $s \in \Lambda(n)$  (2.3) then the multiplicity of the irreducible  $\mathbf{T}(n)$ -module  $L_s(n)$  in the cell representation  $W_t(n)$  (2.2) is one if

$$(1) \quad s = t, \text{ or}$$

(2)  $q^2$  has finite order  $l$ ,  $t + 2l > s > t$  and  $s + t + 2 \equiv 0 \pmod{2l}$ , and zero otherwise.

*Proof.* Adopt the notation of the proof of (5.1). Let  $t \in \Lambda(n)$  and note that  $G_t(n) = G_t^{t+2}(n)$ . If there is no element  $s \in \Lambda(n)$  such that (2) holds, then the computations above show that  $\text{mult}(\det G_t(n)) = 0$ ; hence  $W_t(n)$  is irreducible and the statement follows. If  $q^2$  has finite order  $l$  and  $s \in \Lambda(n)$  satisfies (2), then Corollary (3.5) provides a nonzero homomorphism of  $\mathbf{T}(n)$ -modules  $\theta_n: W_s(n) \rightarrow W_t(n)$ . Hence  $L_s(n)$  is a composition factor of  $W_t(n)$  and we have

$$\dim L_s(n) \leq \dim \text{rad}_t(n) \leq \text{mult}(\det G_t(n))$$

as in the previous proof. However,

$$\dim L_s(n) = \dim W_s(n) - \dim \text{rad}_s(n) \geq \dim W_s(n) - \text{mult}(\det G_s(n)).$$

Now (5.1.2) again asserts that the ends of this chain of inequalities are equal. Therefore we have equality at each step and in particular  $L_s(n)$  is isomorphic to  $\text{rad}_t(n)$ .  $\square$

(5.4) REMARKS.

(1) The decomposition matrices in Theorems (5.1) and (5.3) are “independent of  $n$ ”; one may therefore speak of the multiplicity of  $L_{s,y}$  in  $W_{t,z}$  and of  $L_s$  in  $W_t$ .

(2) Since the dimension of  $W_{t,z}(n)$  is known (1.12), the multiplicities of (5.1) may be used to give formulae for the dimensions of the irreducible modules  $L_{t,z}(n)$ . These formulae are just the inversions of the relations

$$\binom{n}{(n-t)/2} = l_{t,z}(n) + \sum_{\substack{(s,y) \in \Lambda^a \\ (s,y) \succ (t,z)}} l_{s,y}(n)$$

where  $l_{s,y}(n) = \dim L_{s,y}(n)$ . A similar remark applies to the dimensions of the irreducible modules for the Jones and Temperley-Lieb algebras.

(3) The proofs of (5.1) and (5.3) yield the radical series of the modules concerned;  $L_{s,y}(n)$  lies in the  $k$ -th layer of  $W_{t,z}(n)$  if the length of the interval between  $(s, y)$  and  $(t, z)$  in  $\Lambda^a$  is  $k$ . One might expect the layers of the radical series of the cell modules to coincide with the layers (denoted  $\text{rad}^i$  above) of some “Jantzen filtration” of the cell representation and its bilinear form (after scaling the indices).

(4) If the characteristic of  $R$  times the order  $l$  of  $q^2$  exceeds the cardinality of  $n$  then Theorems (5.1) and (5.3) remain valid without the restriction that  $R$  have characteristic zero.

(5) As indicated in (2.9.1), all of our results may be interpreted as statements about the representation theory of  $TL_n^a$ ; in particular, they illuminate a part of the modular representation theory of the affine Hecke algebra  $H_n^a(q)$ . One could ask which irreducible representations of the affine Hecke algebra correspond in the Kazhdan-Lusztig parametrization [KL2] to our  $L_{t,z}$ . A similar comment applies to the connection with the work [Gj].

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