

Zeitschrift: L'Enseignement Mathématique
Herausgeber: Commission Internationale de l'Enseignement Mathématique
Band: 44 (1998)
Heft: 3-4: L'ENSEIGNEMENT MATHÉMATIQUE

Artikel: THE REPRESENTATION THEORY OF AFFINE TEMPERLEY-LIEB ALGEBRAS
Autor: Graham, J. J. / Lehrer, G. I.
Kapitel: §5. DECOMPOSITION MATRICES
DOI: <https://doi.org/10.5169/seals-63902>

Nutzungsbedingungen

Die ETH-Bibliothek ist die Anbieterin der digitalisierten Zeitschriften auf E-Periodica. Sie besitzt keine Urheberrechte an den Zeitschriften und ist nicht verantwortlich für deren Inhalte. Die Rechte liegen in der Regel bei den Herausgebern beziehungsweise den externen Rechteinhabern. Das Veröffentlichen von Bildern in Print- und Online-Publikationen sowie auf Social Media-Kanälen oder Webseiten ist nur mit vorheriger Genehmigung der Rechteinhaber erlaubt. [Mehr erfahren](#)

Conditions d'utilisation

L'ETH Library est le fournisseur des revues numérisées. Elle ne détient aucun droit d'auteur sur les revues et n'est pas responsable de leur contenu. En règle générale, les droits sont détenus par les éditeurs ou les détenteurs de droits externes. La reproduction d'images dans des publications imprimées ou en ligne ainsi que sur des canaux de médias sociaux ou des sites web n'est autorisée qu'avec l'accord préalable des détenteurs des droits. [En savoir plus](#)

Terms of use

The ETH Library is the provider of the digitised journals. It does not own any copyrights to the journals and is not responsible for their content. The rights usually lie with the publishers or the external rights holders. Publishing images in print and online publications, as well as on social media channels or websites, is only permitted with the prior consent of the rights holders. [Find out more](#)

Download PDF: 02.01.2026

ETH-Bibliothek Zürich, E-Periodica, <https://www.e-periodica.ch>

(4.8) COROLLARY.

(1) If n is an odd positive integer, then Jones' annular algebra $\mathbf{J}(n)$ (with parameter $\delta = -q - q^{-1}$) is non-semisimple if and only if there exist distinct odd integers $s, t \in \mathbf{n}$ such that $q^{st} = 1$.

(2) If n is an even positive integer, then Jones' annular algebra $\mathbf{J}(n)$ (with parameter $\delta = -q - q^{-1}$) is non-semisimple if and only if $q^{\frac{n}{2}+1} = 1$ or there exist distinct even integers $s, t \in \mathbf{n}$ such that $q^{\frac{st}{2}} = 1$.

Proof. By [GL, 3.8] the algebra is semisimple precisely when the bilinear pairing $\langle \cdot, \cdot \rangle_{t,z}$ is non-degenerate on each cell representation (of $\mathbf{J}(n)$); this condition is equivalent to the vanishing of the determinant $\det G_{t,z}(n)$, which by (4.7) immediately yields the stated condition. \square

§5. DECOMPOSITION MATRICES

(5.1) THEOREM. Let R be an algebraically closed field of characteristic zero and q a nonzero element of R . Let \preceq be the weakest partial order on the set Λ^a defined in (2.6) such that $(t, z) \preceq (s, y)$ if (t, z) and (s, y) satisfy the hypotheses of Theorem (3.4) for q or q^{-1} . If $(t, z) \in \Lambda^a$, $n \in \mathbf{Z}_{\geq 0}$ and $(s, y) \in \Lambda^a(n)$, then the multiplicity of the irreducible $\mathbf{T}^a(n)$ -module $L_{s,y}(n)$ in the cell representation $W_{t,z}(n)$ of (2.6) is one if $(s, y) \succeq (t, z)$ and zero otherwise.

Proof. Let R be a field and $q \in R$. Let $p: R[y] \rightarrow R$ be the R -algebra homomorphism defined by $y \mapsto q + q^{-1}$, where y is an indeterminate over R . Suppose W is a free $R[y]$ -module of finite rank with an $R[y]$ -bilinear form $\langle \cdot, \cdot \rangle: W \times W \rightarrow R[y]$. If R is regarded as a $R[y]$ -module via the homomorphism p , the free R -module $W_R = R \otimes_{R[y]} W$ inherits an R -bilinear form $\langle \cdot, \cdot \rangle_R: W_R \times W_R \rightarrow R$ given by $\langle 1 \otimes x, 1 \otimes y \rangle_R = p(\langle x, y \rangle)$. Choose $R[y]$ -bases B_1 and B_2 of W and let G denote the associated gram matrix of $\langle \cdot, \cdot \rangle$. If this form is nonsingular (i.e. $\det G \neq 0$), then it may be shown that the multiplicity of the polynomial $y - q - q^{-1}$ in the determinant $\det G$ is greater than or equal to the R -dimension of the radical of $\langle \cdot, \cdot \rangle_R$. In fact if we denote the multiplicity of the polynomial $y - q - q^{-1}$ in $f \in R[y]$ by $\text{mult}(f)$, then

$$\text{mult}(\det G) = \sum_{i \geq 0} \dim \text{rad}^i$$

where rad^i denotes the image under $\phi: W \rightarrow W_R : w \mapsto 1 \otimes w$ of the $R[y]$ -submodule $\{w \in W \mid \langle w, v \rangle \in (y - q - q^{-1})^i R[y] \text{ for any } v \in W\}$.

(Since $R[y]$ is a principal ideal domain, row and column operations may be used to reduce the proof of this fact to the easy case when G is diagonal.) We shall use this elementary result to give a bound for the dimension of the radical of the restriction of $\langle \cdot, \cdot \rangle_{t,z}$ to $W_{t,z}^s(n)$.

Let $t \leq s$ be non-negative integers of the same parity, $n \in \mathbf{Z}_{\geq 0}$ and assume the hypotheses of the statement. Consider $\mathbf{T}_{(R[x], -x)}^a$. We shall compute the determinant of the gram matrix $G_{t,0}^s(n)$ as a polynomial in $y = x + x^{-1}$. Our first goal is to compute the multiplicity of $y - q - q^{-1}$ in this polynomial, i.e. to compute $\text{mult}(\det G_{t,0}^s(n))$. Let l denote the order of q^2 . Since $[n]_x$ and $\begin{bmatrix} n \\ i \end{bmatrix}_x$ are polynomials in $y = x + x^{-1}$ we may speak of the multiplicity of $y - q - q^{-1}$ in these polynomials and it is straightforward that

$$\text{mult}[n]_x = \begin{cases} 1 & \text{if } l \neq 1, \infty \text{ and } l \text{ divides } n, \\ 0 & \text{otherwise,} \end{cases}$$

$$\text{and hence } \text{mult} \begin{bmatrix} n \\ i \end{bmatrix}_x = \begin{cases} 1 & \text{if } l \neq \infty \text{ and } \text{res}_l(n) < \text{res}_l(i), \\ 0 & \text{otherwise,} \end{cases}$$

where $\text{res}_l(n) \in \{0, 1, \dots, l-1\}$ is determined by $\text{res}_l(n) \equiv n \pmod{l}$.

We next give an expression for $\text{mult}([t; r]_x / [s; r]_x)$. Let $r \geq s$ have the same parity as s (or t) and write $X = \{0, 1, \dots, l-1\}$. Then there exist unique elements $k \in \mathbf{Z}$ and $\bar{r} \in X$ such that $r = kl + \bar{r}$. Let \bar{t} denote the unique element of X such that $kl + \bar{t} \equiv \pm t \pmod{2l}$; define \bar{s} similarly. Define:

$$\epsilon_t^s(r) = \begin{cases} 1 & \text{if } \bar{s} \leq \bar{r} < \bar{t}, \\ -1 & \text{if } \bar{t} \leq \bar{r} < \bar{s}, \\ 0 & \text{otherwise.} \end{cases}$$

The function $\epsilon_t^s(r)$ satisfies

- (1) $\epsilon_t^s(r) = \epsilon_s^{-t}(r) = \epsilon_s^{t+2l}(r)$
- (2) $\epsilon_t^s(r) = -\epsilon_s^t(r)$.

It is easy to see that if $0 \leq t \leq s \leq r$, then

$$\epsilon_t^s(r) = \text{mult}([t; r]_x / [s; r]_x).$$

By Corollary (4.5) and Proposition (4.6), we have

$$(5.1.1) \quad \text{mult}(\det G_{t,0}^s(n)) = \sum_{\substack{r \geq s \\ r \equiv t \pmod{2}}} \epsilon_t^s(r) \dim W_r(n).$$

If $l = \infty$ or $s \equiv t$ or $-t \pmod{2l}$, then $\epsilon_t^s(r) = 0$ and so the multiplicity (5.1.1) is zero. For the remainder of this paragraph, assume that $l \neq \infty$ and

$s \not\equiv \pm t \pmod{2l}$. Let $t' \in \mathbf{Z}$ be minimal such that $t' > s$ and $t' \pm t \equiv 0 \pmod{2l}$. Let $s' \in \mathbf{Z}$ be maximal such that $t' > s'$ and $s' \pm s \equiv 0 \pmod{2l}$. Then $s + 2l > t' > s' \geq s > t$. Now in order to compute $\text{mult}(\det G_{t,0}^s(n))$, we partition the sum on the right side of (5.1.1) into three parts:

- (1) $s \leq r < s'$.
- (2) $s' \leq r < t'$.
- (3) $t' \leq r$.

For the terms in the first part, $\epsilon_t^s(r) = 0$. For those in the second part $\epsilon_t^s(r) = 1$ and consequently, these terms contribute $\dim W_{s',0}^{t'}(n) = \sum_{s' \leq r < t'} \dim W_r(n)$ to the sum. The terms in the third part have $\epsilon_t^s(r) = -\epsilon_{s'}^{t'}(r)$ (by properties (1) and (2) of the function $\epsilon_t^s(r)$) and so these terms contribute $\text{mult}(\det G_{s',0}^{t'}(n))$ to the sum.

It follows that

$$(5.1.2) \quad \text{mult}(\det G_{t,0}^s(n)) = \dim W_{s',0}^{t'}(n) - \text{mult}(\det G_{s',0}^{t'}(n)).$$

Note that equation (5.1.2) should be interpreted as a recurrence relation for $\text{mult}(\det G_{t,0}^s(n))$, which together with the initial condition $\text{mult}(\det G_{t,0}^s(n)) = 0$ if $n \leq t$, determines the multiplicity.

Now fix $n \in \mathbf{Z}_{\geq 0}$. Choose $(t, z) \in \Lambda^a$ such that $t \leq n$ and $t \equiv n \pmod{2}$. To prove the Theorem, we shall construct a composition series for $W_{t,z}(n)$.

If (t, z) is maximal in $\Lambda^a(n)$ (with respect to \prec), then it follows from Corollary 4.4 and Proposition 4.6, that $\text{rad}_{t,z}(n) = 0$; hence the irreducible module $L_{t,z}(n)$ coincides with $W_{t,z}(n)$ and the statement follows.

Assume that (t, z) is not a maximal element of $\Lambda^a(n)$ and choose $(s, y) \in \Lambda^a(n)$ such that $(s, y) \succ (t, z)$ and s is minimal with respect to this property. Then the hypotheses of Theorem (3.4) are satisfied (possibly after replacing q by q^{-1}) and so we have an injective homomorphism $\theta_n: W_{s,y}(n) \rightarrow W_{t,z}(n)$ of $\mathbf{T}_{R,q}^a(n)$ -modules. The quotient $Q = W_{t,z}(n)/\text{Im } \theta_n$ has basis $\mu + \text{Im } \theta_n$ indexed by standard diagrams $\mu: t \rightarrow n$ of rank strictly less than $(s-t)/2$. By (2.8), the image of θ_n is contained in $\text{rad}_{t,z}(n)$, whence the bilinear form $\langle \ , \ \rangle_{t,z}$ descends to $Q \times Q \rightarrow R$; its gram matrix (with respect to the basis above) is $G_{t,z}^s(n)$ and $L_{t,z}(n)$ is the quotient of Q by its radical which we denote by $\text{rad}_{t,z}^s(n)$. Consider, for the moment, $\mathbf{T}_{R[x],x}^a$. The multiplicity $\text{mult}(\det G_{t,z}^s(n)) = \text{mult}(\det G_{t,0}^s(n))$ by Corollary (4.4); it follows from the remarks concerning linear algebra at the beginning of this proof that

$$(5.1.3) \quad \dim \text{rad}_{t,z}^s(n) \leq \text{mult}(\det G_{t,0}^s(n)).$$

If the order l (of q^2) is infinite, then (s, y) is the unique element of Λ^a such that $(s, y) \succ (t, z)$. If l is finite and $s \equiv t$ or $-t \pmod{2l}$, then (s, y) is the unique element of Λ^a which covers (t, z) . In either case, we saw above that $\text{mult}(\det G_{t,0}^s(n)) = 0$ and so $\text{rad}_{t,z}^s(n) = 0$. Therefore $Q = L_{t,z}(n)$ and the composition factors of $W_{t,z}(n)$ are $L_{t,z}(n)$ together with those of $W_{s,y}(n)$, as required.

Assume that l is finite and $s \not\equiv \pm t \pmod{2l}$. Let s' and t' be as above and $y' = \epsilon y^{-1}$ where $\epsilon = q^{(s+s')/2} = \pm 1$. Then (s', y') is the unique element of Λ^a such that $(s', y') \succ (t, z)$ and $(s', y') \not\preceq (s, y)$. If $s' > n$, then the initial condition associated with (5.1.2) shows that $\text{mult}(\det G_{t,0}^s(n)) = 0$ and so $\text{rad}_{t,z}^s(n) = 0$; hence $Q = L_{t,z}(n)$ and the statement of (5.1) follows as in the previous paragraph.

Finally, assume that $s' \leq n$. By Theorem (3.4) (with q^{-1} replacing q), there exists an injective $\mathbf{T}^a(n)$ -homomorphism $\theta'_n: W_{s',y'}(n) \rightarrow W_{t,z}(n)$. Thus $L_{s',y'}(n)$ is a composition factor of $W_{t,z}(n)$. Arguing by induction in the poset Λ^a , we may assume that $L_{s',y'}(n)$ is not a composition factor of $W_{s,y}(n) \cong \text{Im}(\theta_n)$ since $(s', y') \not\preceq (s, y)$. It follows that the irreducible module $L_{s',y'}(n)$ is a composition factor of $\text{rad}_{t,z}^s(n)$ and we have, using (5.1.3),

$$\dim L_{s',y'}(n) \leq \dim \text{rad}_{t,z}^s(n) \leq \text{mult}(\det G_{t,0}^s(n)).$$

Arguing as above with (s', y') in place of (t, z) we have

$$\dim L_{s',y'}(n) = \dim Q' - \dim(\text{rad}_{s',y'}^{t'}(n)) \geq \dim W_{s',y'}^{t'}(n) - \text{mult}(\det G_{t',0}^{s'}(n)).$$

Now (5.1.2) asserts that the two ends of this chain of inequalities are equal. Hence we have equality at every step and in particular $L_{s',y'}(n)$ is isomorphic to $\text{rad}_{t,z}^s(n)$. Thus the composition factors of $W_{t,z}(n)$ are $L_{t,z}(n)$ (if $q^2 \neq 0$ or $(t, z) \neq (0, q)$) and $L_{s',y'}(n)$ together with those of $W_{s,y}(n)$, as required. \square

(5.2) COROLLARY. Assume the hypotheses and notation of Theorem 5.1 and let $\mathbf{J}(n)$ be Jones' annular algebra (see (2.10)). If $(t, z) \in \Lambda^a(n)$ is such that $t > 0$ and $z^t = 1$, then the $\mathbf{J}(n)$ -module $W_{t,z}(n)$ has composition factors $L_{s,y}(n)$ indexed by $(s, y) \in \Lambda^a(n)$ such that $(s, y) \succeq (t, z)$. The remaining cell module $W_{0,q}/M$ (2.10) has composition factors $L_{s,y}(n)$ indexed by $(s, y) \in \Lambda^a(n)$ such that $(s, y) \succeq (0, q)$ and $(s, y) \not\preceq (2, 1)$.

The next result is implicit in [DJ] and may be found in [Ma], [GW] and [W].

(5.3) THEOREM. Let R be a field of characteristic zero, let q be a nonzero element of R and let $\mathbf{T}(n) = \mathbf{T}_{R,q}(n)$ be the Temperley-Lieb algebra over R , with parameter q . If $n, t \in \mathbf{Z}_{\geq 0}$ and $s \in \Lambda(n)$ (2.3) then the multiplicity of the irreducible $\mathbf{T}(n)$ -module $L_s(n)$ in the cell representation $W_t(n)$ (2.2) is one if

$$(1) \quad s = t, \text{ or}$$

(2) q^2 has finite order l , $t + 2l > s > t$ and $s + t + 2 \equiv 0 \pmod{2l}$, and zero otherwise.

Proof. Adopt the notation of the proof of (5.1). Let $t \in \Lambda(n)$ and note that $G_t(n) = G_t^{t+2}(n)$. If there is no element $s \in \Lambda(n)$ such that (2) holds, then the computations above show that $\text{mult}(\det G_t(n)) = 0$; hence $W_t(n)$ is irreducible and the statement follows. If q^2 has finite order l and $s \in \Lambda(n)$ satisfies (2), then Corollary (3.5) provides a nonzero homomorphism of $\mathbf{T}(n)$ -modules $\theta_n: W_s(n) \rightarrow W_t(n)$. Hence $L_s(n)$ is a composition factor of $W_t(n)$ and we have

$$\dim L_s(n) \leq \dim \text{rad}_t(n) \leq \text{mult}(\det G_t(n))$$

as in the previous proof. However,

$$\dim L_s(n) = \dim W_s(n) - \dim \text{rad}_s(n) \geq \dim W_s(n) - \text{mult}(\det G_s(n)).$$

Now (5.1.2) again asserts that the ends of this chain of inequalities are equal. Therefore we have equality at each step and in particular $L_s(n)$ is isomorphic to $\text{rad}_t(n)$. \square

(5.4) REMARKS.

(1) The decomposition matrices in Theorems (5.1) and (5.3) are “independent of n ”; one may therefore speak of the multiplicity of $L_{s,y}$ in $W_{t,z}$ and of L_s in W_t .

(2) Since the dimension of $W_{t,z}(n)$ is known (1.12), the multiplicities of (5.1) may be used to give formulae for the dimensions of the irreducible modules $L_{t,z}(n)$. These formulae are just the inversions of the relations

$$\binom{n}{(n-t)/2} = l_{t,z}(n) + \sum_{\substack{(s,y) \in \Lambda^a \\ (s,y) \succ (t,z)}} l_{s,y}(n)$$

where $l_{s,y}(n) = \dim L_{s,y}(n)$. A similar remark applies to the dimensions of the irreducible modules for the Jones and Temperley-Lieb algebras.

(3) The proofs of (5.1) and (5.3) yield the radical series of the modules concerned; $L_{s,y}(n)$ lies in the k -th layer of $W_{t,z}(n)$ if the length of the interval between (s, y) and (t, z) in Λ^a is k . One might expect the layers of the radical series of the cell modules to coincide with the layers (denoted rad^i above) of some “Jantzen filtration” of the cell representation and its bilinear form (after scaling the indices).

(4) If the characteristic of R times the order l of q^2 exceeds the cardinality of n then Theorems (5.1) and (5.3) remain valid without the restriction that R have characteristic zero.

(5) As indicated in (2.9.1), all of our results may be interpreted as statements about the representation theory of TL_n^a ; in particular, they illuminate a part of the modular representation theory of the affine Hecke algebra $H_n^a(q)$. One could ask which irreducible representations of the affine Hecke algebra correspond in the Kazhdan-Lusztig parametrization [KL2] to our $L_{t,z}$. A similar comment applies to the connection with the work [Gj].

REFERENCES

- [Ch] CHEREDNIK, I. V. A new interpretation of Gel'fand Tzvetlin bases. *Duke Math. J.* 54 (1987), 563–577.
- [DJ] DIPPER, R. and G. JAMES. The q -Schur algebra. *Proc. London Math. Soc.* (3) 59 (1989), no. 1, 23–50.
- [FG] FAN, C. K. and R. M. GREEN. On the affine Temperley-Lieb algebras. Preprint.
- [FY] FREYD, P. J. and D. N. YETTER. Braided compact closed categories with applications to low-dimensional topology. *Adv. Math.* 77 (1989), 156–182.
- [Gj] GROJNOWSKI, I. Representations of affine Hecke algebras (and affine quantum GL_n) at roots of unity. *Internat. Math. Res. Notices*, 1994, no. 5, 215ff.
- [GL] GRAHAM, J. J. and G. I. LEHRER. Cellular Algebras. *Invent. Math.* 123 (1996), 1–34.
- [Gr] GRAHAM, J. J. PhD Thesis. Sydney University, 1995.
- [GW] GOODMAN, F. M. and H. WENZL. The Temperley-Lieb algebra at roots of unity. *Pacific J. Math.* 161 (1993), 307–334.
- [J1] JONES, V. F. R. Hecke algebra representations of braid groups and link polynomials. *Ann. Math.* 126, (1987), 335–388.
- [J2] ——— A quotient of the affine Hecke algebra in the Brauer algebra. *L'Enseignement Math.* (2) 40 (1994), 313–344.
- [J3] ——— Index for subfactors. *Invent. Math.* 72 (1983), 1–25.
- [Ja] JAMES, G. *Representations of General Linear Groups*. London Math. Soc. Lect. Note Series 94. Cambridge University Press, Cambridge, 1984.
- [KL1] KAZHDAN, D. and G. LUSZTIG. Equivariant K -theory and representations of Hecke algebras. *Proc. Amer. Math. Soc.* 94 (1985), 337–342.