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COROLLARY 3.2. *For  $\#X = k$ ,  $\#R = n$ ,  $x_0 \in X$  and  $0 < \epsilon < 1/k$  fixed, being  $(\epsilon, x_0)$ -balanced is generic for  $\Gamma = \langle X | R \rangle$ .*

*Proof of corollary.* We choose  $n$  relations at random; by Lemma 3.1, every  $r \in R$  is generically  $(\epsilon, x_0)$ -balanced, but the conjunction of finitely many generic properties is also generic.  $\square$

#### 4. SOME SUFFICIENT CONDITIONS FOR THE EXISTENCE OF FREE SUBGROUPS

We first begin by a very easy proposition.

PROPOSITION 4.1. *Let  $\Gamma = \langle X | R \rangle$  be a finite presentation, which has a Dehn algorithm and such that for some  $y \in X$  every subword  $u$  of every  $r \in R^*$  with  $|u| > |r|/2$  contains either  $y$  or  $y^{-1}$ , then  $X - \{y\}$  generates a free subgroup in  $\Gamma$ .*

The proof of this proposition will follow from Lemma 4.2 below.

LEMMA 4.2. *For  $\langle X | R \rangle$  a finite presentation of a group  $\Gamma$  and  $y \in X$ , the following are equivalent:*

- $X - \{y\}$  freely generates a free subgroup of  $\Gamma$ ;
- every non trivial element  $\omega \in \mathbf{F}_X$ , which represents the identity in  $\Gamma$ , contains either  $y$  or  $y^{-1}$ .

*Proof.* 1)  $\Rightarrow$  2): By contraposition, suppose that there exists a non trivial reduced element  $\omega \in \mathbf{F}_{X-\{y\}}$  such that  $\bar{\omega} = e$  (where  $\bar{\omega}$  is the canonical projection of  $\omega$  in  $\Gamma$ ), then  $X - \{y\}$  does not freely generate a free subgroup in  $\Gamma$ .

2)  $\Rightarrow$  1): Let  $\omega_1, \omega_2 \in \mathbf{F}_{X-\{y\}}$  be two reduced elements such that  $\bar{\omega}_1 = \bar{\omega}_2 \in \Gamma$ . Then  $\overline{\omega_1 \omega_2^{-1}} = e \in \Gamma$ . So  $\omega_1 \omega_2^{-1}$  is an element of  $\mathbf{F}_{X-\{y\}}$  which represents the identity in  $\Gamma$ . By hypothesis, this implies  $\omega_1 = \omega_2$  in  $\mathbf{F}_X$ . Hence  $X - \{y\}$  freely generates a free subgroup in  $\Gamma$ .  $\square$

*Proof of Proposition 4.1.* By Lemma 4.2, it is sufficient to show that every non trivial reduced word on  $\mathbf{F}_X$  which represents the identity in  $\Gamma$  contains either  $y$  or  $y^{-1}$ . By assumption,  $\Gamma = \langle X | R \rangle$  satisfies a Dehn algorithm, so such a word contains at least one half of a relator  $r$  in  $R$  which contains at least one occurrence of  $y$  or  $y^{-1}$ .  $\square$

The interest of this proposition appears when we replace “having a Dehn’s algorithm” by “satisfying the small cancellation condition  $C'(1/6)$ ”, because  $C'(1/6)$  and the fact that every subword  $u$  of any relation  $r$  with  $|u| > |r|/2$  contains at least one  $y$  or  $y^{-1}$  are easy to check on a given presentation.

Unfortunately, as explained before, it is not known if the small cancellation hypothesis is generic, so we need other sufficient conditions to ensure that  $X - \{y\}$  generates a free subgroup in  $\Gamma$ .

**PROPOSITION 4.3.** *Let  $\Gamma = \langle X \mid R \rangle$  be a finite presentation with  $k$  generators and  $l$  relations, which is  $(\epsilon, x_0)$ -balanced for some  $0 < \epsilon < 1/k$  and some  $x_0 \in X$ , and which satisfies a  $\theta$ -condition such that  $\theta < \epsilon/(2 - \epsilon)$ . Then  $X - \{x_0\}$  freely generates a free group in  $\Gamma$ .*

To prove the proposition we need the following lemma and the following notations. For a cell  $f_i$  of the diagram, we denote by  $Int(f_i)$  (resp. by  $Ext(f_i)$ ) the number of edges of  $f_i$  which are internal to the diagram (resp. which are on the border of the diagram). We denote also by  $\#(f_i)$  the total number of edges of the cell  $f_i$ .

**LEMMA 4.4.** *Let  $\Gamma = \langle X \mid R \rangle$  be a finite presentation of a group  $\Gamma$  which satisfies a  $\theta$ -condition for some  $0 < \theta < 1$ , then for every reduced diagram, there exists a 2-cell  $f$  of  $\Delta$  satisfying*

$$Int(f) \leq \frac{2\theta}{1 + \theta} \#(f).$$

*Proof.* First we prove it for simple diagrams. Let  $\epsilon = 2\theta/(1 + \theta)$ . Because the diagram is simple we have the following equalities:

- I)  $\sum_i Ext(f_i) = E(\Delta) = |\partial\Delta|,$
- II)  $\sum_i Int(f_i) = 2I(\Delta)$ , because every internal edge belongs to two different cells.

So we get:

$$\#(\Delta) = \frac{1}{2} \sum_i Int(f_i) + \sum_i Ext(f_i) = \sum_i \#(f_i) - \frac{1}{2} \sum_i Int(f_i).$$

To obtain a contradiction, we suppose that every cell  $f_i$  of one diagram  $\Delta$  is such that  $(1/\epsilon)Int(f_i) > \#(f_i)$ . Then we have

$$\frac{1}{\epsilon} \sum_i \text{Int}(f_i) > \sum_i \#(f_i) = \#(\Delta) + \frac{1}{2} \sum_i \text{Int}(f_i),$$

whence  $\frac{2-\epsilon}{2\epsilon} \sum_i \text{Int}(f_i) > \#(\Delta)$  or  $\frac{2-\epsilon}{\epsilon} I(\Delta) > \#(\Delta)$ . Since  $\epsilon = 2\theta/(1+\theta)$ , we obtain  $I(\Delta) > \theta\#(\Delta)$ , which contradicts the  $\theta$ -condition.

In fact, if the reduced diagram  $\Delta$  is not simple, it is a union of simple diagrams linked by bridges. So each of its parts, which is a simple diagram, defines another reduced diagram (relative to another word), so the inequality holds for every part of  $\Delta$  which is a simple diagram. We conclude by saying that increasing the number of external edges does not affect the inequality.  $\square$

*Proof of 4.3.* By Lemma 4.2, it is sufficient to prove that the  $(\epsilon, x_0)$ -balanced and  $\theta$ -conditions imply that every non trivial reduced word in  $\mathbf{F}_X$  which vanishes in  $\Gamma$  contains at least one  $x_0^{\pm 1}$ .

Let us choose such a word  $\omega$  and  $\Delta$  a reduced diagram of  $\omega$ . By Lemma 4.4, there exists a cell  $f$  with border equal to one  $r \in R^*$ , such that

$$\text{Int}(f) \leq \frac{2\theta}{1+\theta} \#(f) = \frac{2\theta}{1+\theta} |r| < \epsilon |r| \leq n_{x_0}(r),$$

because  $\theta < \epsilon/(2-\epsilon)$ . As there are more occurrences of  $x_0$  or  $x_0^{-1}$  than the number of internal edges, it means that some occurrences of  $x_0$  or  $x_0^{-1}$  will be external edges, i.e. will be in the border of  $\Delta$  which is  $\omega$ .  $\square$

We are now able to prove the main theorem.

*Proof of theorem 1.1.* By Proposition 4.3, for a finite presentation  $\langle X | R \rangle$ , we know that being  $(\epsilon, x_0)$ -balanced and satisfying a  $\theta$ -condition is sufficient to ensure that  $X - \{x_0\}$  freely generates a free subgroup in  $\Gamma$ . But by Corollary 3.2 and [13, Theorem 2], these two conditions are generic and so is the conjunction of these two conditions.  $\square$

## 5. SPECTRAL ESTIMATES FOR ADJACENCY OPERATORS ON CAYLEY GRAPHS

The existence of a free subgroup generated by  $X - \{x_0\}$  gives an upper bound for the spectral value of the adjacency operator on the Cayley graph of  $\Gamma = \langle X | R \rangle$  associated with the symmetric generating system  $S = X \cup X^{-1}$ .

We briefly recall some definitions and notations. The Cayley graph  $G(\Gamma, X)$  of  $\Gamma$  associated with  $S$  has its set of vertices in bijection with  $\Gamma$  and two