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Autor: Hausmann, Jean-Claude / Knutson, Allen
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section collapsed. The trace function on $\mathcal{M}_{m \times 2}(\mathbf{C})$ descends to $\tilde{\mathbf{G}}_2(\mathbf{C}^m)$ and to the Casimir function “perimeter” on ${}^m\mathcal{PP}_+^3$.

4. POLYGONS WITH GIVEN SIDES – KÄHLER STRUCTURES

We now use the map $\ell : {}^m\tilde{\mathcal{P}}^k, {}^m\mathcal{P}_+^k, {}^m\mathcal{P}^k \rightarrow \mathbf{R}^m$ defined in (2.4). Recall that $\ell(\rho)$, for $\rho \in {}^m\tilde{\mathcal{P}}^k$, is the length of the successive sides of a representative of r with total perimeter 2.

For $\alpha = (\alpha_1, \dots, \alpha_m) \in \mathbf{R}_{\geq 0}^m$ with $\sum_{i=1}^m \alpha_i = 2$, we define

$${}^m\tilde{\mathcal{P}}^k(\alpha) := \tilde{\mathcal{P}}^k(\alpha) := \{\rho \in {}^m\tilde{\mathcal{P}}^k \mid \ell(\rho) = \alpha\} \subset {}^m\tilde{\mathcal{P}}^k.$$

The space $\tilde{\mathcal{P}}^k(\alpha)$ is invariant under the action of O_k . We define the moduli spaces

$$\mathcal{P}_+^k(\alpha) := SO_k \backslash \tilde{\mathcal{P}}^k(\alpha) = \ell^{-1}(\alpha) \subset {}^m\mathcal{P}_+^k$$

and

$$\mathcal{P}^k(\alpha) := O_k \backslash \tilde{\mathcal{P}}^k(\alpha) = \ell^{-1}(\alpha) \subset {}^m\mathcal{P}^k.$$

The space $\tilde{\mathcal{P}}^1(\alpha)$ consists of a finite number of points and is generically empty. We call α *generic* if $\tilde{\mathcal{P}}^1(\alpha) = \emptyset$.

THEOREM 4.1. *The map $\mu := \ell \circ \hat{\Phi} : \mathbf{G}_2(\mathbf{C}^m) \longrightarrow \mathbf{R}^m$ is a moment map for the action of U_1^m on $\mathbf{G}_2(\mathbf{C}^m)$.*

Proof. As seen in (3.13), the moment map $\Psi : \mathbf{G}_2(\mathbf{C}^m) \longrightarrow \mathcal{H}(m)$ for the U_m -action on $\mathbf{G}_2(\mathbf{C}^m)$ is induced from $\tilde{\Psi} : \mathcal{M}_{m \times 2}(\mathbf{C}) \longrightarrow \mathcal{H}(m)$ given by $\tilde{\Psi}(a, b) := (a, b) \cdot (a, b)^*$. A moment map μ for the action of U_1^m is obtained by composing Ψ with the projection $\mathcal{H}(m) \longrightarrow \mathbf{R}^m$ associating to a matrix its diagonal entries. So, if $\Pi \in \mathbf{G}_2(\mathbf{C}^m)$ is generated by a and b with $(a, b) \in \mathbf{V}_2(\mathbf{C}^m)$, one has

$$\mu(\Pi) = (|a_1|^2 + |b_1|^2, \dots, |a_m|^2 + |b_m|^2) = \ell \circ \hat{\Phi}(a, b). \quad \square$$

A now classic theorem of Atiyah and Guillemin-Sternberg [Au, §III.4.2] asserts that the image of a moment map for a torus action is a convex polytope (the *moment polytope*). The restriction of the moment map to the fixed point set of an anti-symplectic involution has the same image [Du]. In our case, one gets these facts directly:

COROLLARY 4.2. *The moment map $\mu: \mathbf{G}_2(\mathbf{C}^m) \longrightarrow \mathbf{R}^m$ satisfies $\mu(\mathbf{G}_2(\mathbf{C}^m)) = \mu(\mathbf{G}_2(\mathbf{R}^m)) = \Xi_m$, where Ξ_m is the hypersimplex*

$$\Xi_m := \{(x_1, \dots, x_m) \in \mathbf{R}^m \mid 0 \leq x_i \leq 1 \text{ and } \sum_{i=1}^m x_i = 2\}.$$

Proof. One has $\text{Image}(\mu) = \text{Image}(\ell)$. Further it is manifest that $\text{Image}(\ell) \subset \Xi_m$. A proof that $\text{Image}(\ell) = \Xi_m$ is actually provided in [KM1], Lemma 1, or [Ha]. We give here however another argument, for the pleasure of constructing a continuous section $\sigma: \Xi_m \longrightarrow {}^m\mathcal{P}^2$ of ℓ . If $m = 3$, we have already mentioned in (2.7) that ${}^3\mathcal{P}^2$ is homeomorphic to Ξ_3 via the map ℓ . Let $\alpha \in \Xi_m$. Define $\beta_i := \sum_{j=1}^i \alpha_j$ and

$$r(\alpha) := \min\{i \mid \beta_i \leq 1 \text{ and } \beta_{i+1} \geq 1\}.$$

The numbers $\beta_r, \alpha_r, 2 - \beta_{r+1}$ form a triple of Ξ_3 and are then the lengths of a unique triangle $\tau(\alpha) \in {}^3\mathcal{P}^2$, which can be subdivided in the obvious way to define the element $\sigma(\alpha) \in {}^m\mathcal{P}^2(\alpha)$ (see Figure 1).

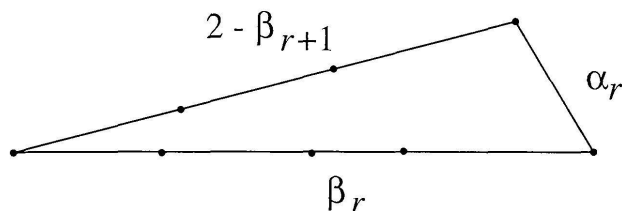


FIGURE 1: $\tau(\alpha)$

The continuity of σ comes from the fact that if the map r is discontinuous at some α , the triangle $\tau(\alpha)$ is then lined. \square

REMARKS. 1) Corollary 4.2 is also a consequence of our stronger result (5.4).

2) The word “hypersimplex” is introduced in [GM]. Observe that H is obtained by taking the convex hull of the middle point of each edge of a standard $(m - 1)$ -simplex.

We also obtain the critical values of μ (compare [Ha]):

PROPOSITION 4.3. *The set of critical values of μ on $\mathbf{G}_2(\mathbf{C}^m) \rightarrow \Xi_m$ or $\mathbf{G}_2(\mathbf{R}^m) \rightarrow \Xi_m$ consists of those points $(x_1, \dots, x_m) \in \Xi_m$ satisfying one of the following conditions:*

- a) *one x_i vanishes;*
- b) *one x_i is equal to 1;*
- c) *there exist $\varepsilon_i = \pm 1$ such that $\sum_{i=1}^m \varepsilon_i x_i = 0$, with at least two ε_i 's of each sign.*

REMARK. Points satisfying a) and b) constitute the boundary of Ξ_m . Points satisfying c) are “inner walls”. Points satisfying a) correspond to non-proper polygons. Those satisfying b) or c) are non-generic α 's (Condition b) implies that there exist $\varepsilon_i = \pm 1$ such that $\sum_{i=1}^m \varepsilon_i x_i = 0$ with all but one ε_i of the same sign.)

Proof. The critical points of the moment map μ are the points of $\mathbf{G}_2(\mathbf{C}^m)$ for which the U_1^m -action has a stabilizer of dimension bigger than 1. They are the images of those $(2 \times m)$ -matrices in $\mathbf{V}_2(\mathbf{C}^m)$ for which the $(U_1^m \times_{U_1} U_2)$ -action has a non-discrete stabilizer. There are such points whose stabilizer is contained in $U_1^m \times \{1\}$; they are the matrix with one row vanishing and their values under μ are the points of Ξ_m satisfying a). The other points give rise to points in ${}^m\tilde{\mathcal{P}}^3 = U_1^m / \mathbf{V}_2(\mathbf{C}^m)$ so that the action of $U_2 / \{\text{center of } U_2\} \simeq SO_3$ has non discrete stabilizer. Those points are the lined configurations ${}^m\tilde{\mathcal{P}}^1$. Their values in Ξ_m are the non generic α 's, which are the points in Ξ_m satisfying b) or c). \square

We have proven most of the main result of this section: for generic and proper α , the space $\mathcal{P}^3(\alpha)$ is a Kähler sub-quotient of $\mathbf{G}_2(\mathbf{C}^m)$.

THEOREM 4.4. *For $\alpha \in \text{int } \Xi_m$ generic, $\mathcal{P}_+^3(\alpha)$ is a Kähler manifold isomorphic to the Kähler reduction $U_1^m \backslash \mu^{-1}(\alpha)$. The involution \smile is antiholomorphic and $\mathcal{P}^2(\alpha)$ can be seen as the real part of $\mathcal{P}_+^3(\alpha)$.*

Proof. By 4.1, one has $\mathcal{P}^3(\alpha) = \ell^{-1}(\alpha) = U_1^m \backslash \mu^{-1}(\alpha)$ and we have seen in 3.9 that $\widehat{\Phi}(\bar{a}, \bar{b}) = \Phi(a, b)^\smile$. \square

We shall now compare the Kähler structure obtained on $\mathcal{P}_+^3(\alpha)$ from the Grassmannian to that introduced by Klyachko [Kl] or Kapovich-Millson ([KM2], §3). Using the standard cross product \times and scalar product $\langle \cdot, \cdot \rangle$ on \mathbf{R}^3 , these authors put on the sphere S_r^2 of radius r the complex structure \tilde{J}

defined by

$$\tilde{J}v := \frac{1}{r} x \times v \quad (v \in T_x S_r^2)$$

and the Kähler metric

$$\tilde{h}(u, v) := \frac{1}{r} \langle u, v \rangle - \frac{i}{r^2} \langle x, u \times v \rangle \quad (u, v \in T_x S_r^2)$$

with associated symplectic form $\tilde{\omega}(u, v) := \langle \frac{x}{r^2} u \times v \rangle$. Let $W(\alpha) := \prod_{i=1}^m S_{\alpha_i}^2$. The map $\beta : W_\alpha \rightarrow \mathbf{R}^3$ defined by $\beta(z_1, \dots, z_m) := \sum_{i=1}^m z_i$ is the moment map for the diagonal action of SO_3 on W_α . The space $\mathcal{P}_+^3(\alpha)$ thus occurs as the symplectic reduction $SO_3 \backslash \beta^{-1}(0)$.

PROPOSITION 4.5. *The complex structure J and Kähler metric h of 4.4 compare with those \tilde{J} and \tilde{h} of Kapovich-Millson in the following way:*

$$\tilde{J} = J \quad \text{and} \quad \tilde{h}(u, v) = 4h(u, v).$$

Proof. Starting from the Hermitian vector space $\mathcal{M} = \mathcal{M}_{m \times 2}(\mathbf{C})$ one sees that $\mathcal{P}^3(\alpha)$ is obtained by two successive symplectic reductions

$$\mathbf{G}_2(\mathbf{C}^m) = \tilde{\Phi}^{-1}(0)/U_2 \quad \text{and} \quad \mathcal{P}^3(\alpha) = U_1^m \backslash \mu^{-1}(\alpha)$$

(we use the notation of §3). One can perform the reductions in the reverse order. We first get

$$U_1^m \backslash \tilde{\Psi}^{-1}(\alpha) = \prod_{i=1}^m \mathbf{C}P_{\alpha_i}^1$$

where $\mathbf{C}P_r^1$ is the quotient of the 3-dimensional sphere

$$\{(u, v) \in \mathbf{C}^2 \mid |u|^2 + |v|^2 = r\}$$

by the diagonal action of U_1 . The moment map $\tilde{\Phi} : \mathcal{M} \rightarrow \mathcal{H}(2)$ gives a moment map (still called $\tilde{\Phi}$) from the product of projective spaces into $\mathcal{H}_0(2)$. One has a commutative diagram

$$\begin{array}{ccc} \prod_{i=1}^m \mathbf{C}P_{\alpha_i}^1 & \xrightarrow[\simeq]{\Pi^\phi} & \prod_{i=1}^m S_{\alpha_i}^2 \\ \tilde{\Phi} \downarrow & & \downarrow \beta \\ \mathcal{H}_0(2) & \xrightarrow[\simeq]{\psi} & \mathbf{R}^3 \end{array}$$

where $\psi : \mathcal{H}_0(2) \rightarrow \mathbf{R}^3 \simeq \mathbf{R} \times \mathbf{C}$ sends the matrix $\begin{pmatrix} u & z \\ \bar{z} & -u \end{pmatrix}$ to (u, z) .

To prove Proposition 4.5, it is enough to establish that for all $a \in \mathbf{CP}_r^1$, the tangent map $T_a\phi : T_a\mathbf{CP}_r^1 \longrightarrow T_{\phi(a)}S_r^2$ satisfies

$$T_a\phi(Jv) = \tilde{J}T_a\phi(v) \quad \text{and} \quad \tilde{\omega}(T_a\phi(v), T_a\phi(Jv)) = 4\omega(v, Jv).$$

By U_2 -equivariance, we can restrict ourselves to $a = [\sqrt{r}, 0]$. The tangent space $T_a\mathbf{CP}_r^1$ is identified with $\{0\} \times \mathbf{C}$ and one can take $v = (0, 1)$ and $Jv = (0, i)$. One has $\phi(a) = (r, 0, 0)$,

$$T_a\phi(v) = (0, 2\sqrt{r}, 0), \quad T_a\phi(Jv) = (0, 0, 2\sqrt{r}) = \tilde{J}T_a\phi(v)$$

and $\tilde{\omega}(T_a\phi(v), T_a\phi(Jv)) = 4$, while $\omega(v, Jv) = 1$. \square

REMARKS

(4.6) The results of this section show that the spaces $\mathcal{P}_+^3(\alpha)$ for generic α are the symplectic leaves of the Poisson structure on the regular part of ${}^m\mathcal{P}_+^3$, or ${}^m\mathcal{PP}_+^3$ given in (3.13) and (3.14).

(4.7) If one works in the pure quaternions $I\mathbf{H}$, the complex structure \tilde{J} on S_r^2 becomes

$$\tilde{J}(v) = \frac{qv}{|q|}, \quad (v \in T_q S_r^2 = I\mathbf{H}).$$

The sphere S_r^2 is a co-adjoint orbit of $U_1(\mathbf{H})$ and the Hermitian form $\tilde{\omega}$ is the Kirillov–Kostant form (see [Gu, Theorem 1.1]).

(4.8) The isomorphism between the symplectic reductions of the Grassmannian $\mathbf{G}_2(\mathbf{C}^m)$ and the product of \mathbf{CP}^1 's that underlies our results 3.9, 4.4 and the proof of 4.5 is a symplectic version of the Gel'fand–MacPherson correspondence ([GM] and [GGMS]). The fact that this isomorphism comes from two reductions of \mathcal{M} is the philosophy of “dual pairs” (see [Mo] and the references therein).

5. THE GEL'FAND–CETLIN ACTION

On ${}^m\mathcal{F}^k$ we have so far defined the length functions $\tilde{\ell}$ measuring the distances between successive vertices. We now introduce $\tilde{d} : {}^m\mathcal{F}^k \rightarrow \mathbf{R}^m$, $\tilde{d}(\rho) = (|\rho(1)|, |\rho(1) + \rho(2)|, \dots, |\sum_{i=1}^m \rho(i)|)$, the lengths of the diagonals connecting the vertices to the origin. (Only $m-3$ of these functions are new, as $\tilde{d}(\rho)_1 = \tilde{\ell}(\rho)_1$, $\tilde{d}(\rho)_{m-1} = \tilde{\ell}(\rho)_m$, and $\tilde{d}(\rho)_m = 0$. Hereafter we write only ℓ_i, d_i and the ρ is to be understood.)