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section collapsed. The trace function on $\mathcal{M}_{m\times 2}(\mathbf{C})$ descends to $\widetilde{\mathbf{G}}_2(\mathbf{C}^m)$ and to the Casimir function "perimeter" on ${}^m\mathcal{PP}^3_+$.

4. Polygons with given sides – Kähler structures

We now use the map $\ell : {}^m\widetilde{\mathcal{P}}^k, {}^m\mathcal{P}^k_+, {}^m\mathcal{P}^k \to \mathbf{R}^m$ defined in (2.4). Recall that $\ell(\rho)$, for $\rho \in {}^m\widetilde{\mathcal{P}}^k$, is the length of the successive sides of a representative of r with total perimeter 2.

For $\alpha = (\alpha_1, \ldots, \alpha_m) \in \mathbf{R}^m_{\geq 0}$ with $\sum_{i=1}^m \alpha_i = 2$, we define

$$ {}^m\widetilde{\mathcal{P}}^k(\alpha) =: \widetilde{\mathcal{P}}^k(\alpha) := \{\rho \in {}^m\widetilde{\mathcal{P}}^k \mid \ell(\rho) = \alpha\} \subset {}^m\widetilde{\mathcal{P}}^k . $$

The space $\widetilde{\mathcal{P}}^k(\alpha)$ is invariant under the action of O_k. We define the moduli spaces

$$ \mathcal{P}^k_+(\alpha) := SO_k \backslash \widetilde{\mathcal{P}}^k(\alpha) = \ell^{-1}(\alpha) \subset {}^m\mathcal{P}^k_+ $$

and

$$ \mathcal{P}^k(\alpha) := O_k \backslash \widetilde{\mathcal{P}}^k(\alpha) = \ell^{-1}(\alpha) \subset {}^m\mathcal{P}^k . $$

The space $\widetilde{\mathcal{P}}^1(\alpha)$ consists of a finite number of points and is generically empty. We call α *generic* if $\widetilde{\mathcal{P}}^1(\alpha) = \varnothing$.

THEOREM 4.1. *The map $\mu := \ell \circ \widehat{\Phi} : \mathbf{G}_2(\mathbf{C}^m) \longrightarrow \mathbf{R}^m$ is a moment map for the action of U_1^m on $\mathbf{G}_2(\mathbf{C}^m)$.*

Proof. As seen in (3.13), the moment map $\Psi : \mathbf{G}_2(\mathbf{C}^m) \longrightarrow \mathcal{H}(m)$ for the U_m-action on $\mathbf{G}_2(\mathbf{C}^m)$ is induced from $\widetilde{\Psi} : \mathcal{M}_{m\times 2}(\mathbf{C}) \longrightarrow \mathcal{H}(m)$ given by $\widetilde{\Psi}(a, b) := (a, b) \cdot (a, b)^*$. A moment map μ for the action of U_1^m is obtained by composing Ψ with the projection $\mathcal{H}(m) \longrightarrow \mathbf{R}^m$ associating to a matrix its diagonal entries. So, if $\Pi \in \mathbf{G}_2(\mathbf{C}^m)$ is generated by a and b with $(a, b) \in \mathbf{V}_2(\mathbf{C}^m)$, one has

$$ \mu(\Pi) = (|a_1|^2 + |b_1|^2, \ldots, |a_m|^2 + |b_m|^2) = \ell \circ \widehat{\Phi}(a, b) . \qquad \square $$

A now classic theorem of Atiyah and Guillemin-Sternberg [Au, § III.4.2] asserts that the image of a moment map for a torus action is a convex polytope (the *moment polytope*). The restriction of the moment map to the fixed point set of an anti-symplectic involution has the same image [Du]. In our case, one gets these facts directly:

COROLLARY 4.2. *The moment map* $\mu\colon \mathbf{G}_2(\mathbf{C}^m) \longrightarrow \mathbf{R}^m$ *satisfies* $\mu\big(\mathbf{G}_2(\mathbf{C}^m)\big) = \mu\big(\mathbf{G}_2(\mathbf{R}^m)\big) = \Xi_m$, *where* Ξ_m *is the hypersimplex*

$$\Xi_m := \{(x_1, \dots x_m) \in \mathbf{R}^m \mid 0 \le x_i \le 1 \quad \textit{and} \quad \sum_{i=1}^m x_i = 2\}\,.$$

Proof. One has $\mathrm{Image}(\mu) = \mathrm{Image}(\ell)$. Further it is manifest that $\mathrm{Image}(\ell) \subset \Xi_m$. A proof that $\mathrm{Image}(\ell) = \Xi_m$ is actually provided in [KM1], Lemma 1, or [Ha]. We give here however another argument, for the pleasure of constructing a continuous section $\sigma : \Xi_m \longrightarrow {}^m\mathcal{P}^2$ of ℓ. If $m = 3$, we have already mentioned in (2.7) that ${}^3\mathcal{P}^2$ is homeomorphic to Ξ_3 via the map ℓ. Let $\alpha \in \Xi_m$. Define $\beta_i := \sum_{j=1}^i \alpha_j$ and

$$r(\alpha) := \min\{i \mid \beta_i \le 1 \quad \text{and} \quad \beta_{i+1} \ge 1\}\,.$$

The numbers $\beta_r, \alpha_r, 2-\beta_{r+1}$ form a triple of Ξ_3 and are then the lengths of a unique triangle $\tau(\alpha) \in {}^3\mathcal{P}^2$, which can be subdivided in the obvious way to define the element $\sigma(\alpha) \in {}^m\mathcal{P}^2(\alpha)$ (see Figure 1).

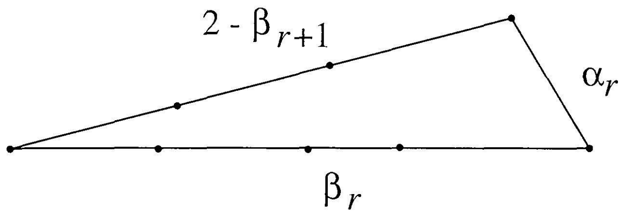

FIGURE 1: $\tau(\alpha)$

The continuity of σ comes from the fact that if the map r is discontinuous at some α, the triangle $\tau(\alpha)$ is then lined. □

REMARKS. 1) Corollary 4.2 is also a consequence of our stronger result (5.4).

2) The word "hypersimplex" is introduced in [GM]. Observe that H is obtained by taking the convex hull of the middle point of each edge of a standard $(m-1)$-simplex.

We also obtain the critical values of μ (compare [Ha]):

PROPOSITION 4.3. *The set of critical values of* μ *on* $\mathbf{G}_2(\mathbf{C}^m) \to \Xi_m$ *or* $\mathbf{G}_2(\mathbf{R}^m) \to \Xi_m$ *consists of those points* $(x_1, \dots x_m) \in \Xi_m$ *satisfying one of the following conditions:*

a) one x_i *vanishes;*

b) one x_i *is equal to* 1 *;*

c) there exist $\varepsilon_i = \pm 1$ *such that* $\sum_{i=1}^m \varepsilon_i x_i = 0$, *with at least two* ε_i*'s of each sign.*

REMARK. Points satisfying a) and b) constitute the boundary of Ξ_m. Points satisfying c) are "inner walls". Points satisfying a) correspond to non-proper polygons. Those satisfying b) or c) are non-generic α's (Condition b) implies that there exist $\varepsilon_i = \pm 1$ such that $\sum_{i=1}^m \varepsilon_i x_i = 0$ with all but one ε_i of the same sign.)

Proof. The critical points of the moment map μ are the points of $\mathbf{G}_2(\mathbf{C}^m)$ for which the U_1^m-action has a stabilizer of dimension bigger than 1. They are the images of those $(2 \times m)$-matrices in $\mathbf{V}_2(\mathbf{C}^m)$ for which the $(U_1^m \times_{U_1} U_2)$-action has a non-discrete stabilizer. There are such points whose stabilizer is contained in $U_1^m \times \{1\}$; they are the matrix with one row vanishing and their values under μ are the points of Ξ_m satisfying a). The other points give rise to points in ${}^m\widetilde{\mathcal{P}}^3 = U_1^m / \mathbf{V}_2(\mathbf{C}^m)$ so that the action of $U_2/\{\text{center of } U_2\} \simeq SO_3$ has non discrete stabilizer. Those points are the lined configurations ${}^m\widetilde{\mathcal{P}}^1$. Their values in Ξ_m are the non generic α's, which are the points in Ξ_m satisfying b) or c). $\square$

We have proven most of the main result of this section: for generic and proper α, the space $\mathcal{P}^3(\alpha)$ is a Kähler sub-quotient of $\mathbf{G}_2(\mathbf{C}^m)$.

THEOREM 4.4. *For* $\alpha \in \operatorname{int} \Xi_m$ *generic,* $\mathcal{P}^3_+(\alpha)$ *is a Kähler manifold isomorphic to the Kähler reduction* $U_1^m \backslash \mu^{-1}(\alpha)$. *The involution* $\check{}$ *is antiholomorphic and* $\mathcal{P}^2(\alpha)$ *can be seen as the real part of* $\mathcal{P}^3_+(\alpha)$.

Proof. By 4.1, one has $\mathcal{P}^3(\alpha) = \ell^{-1}(\alpha) = U_1^m \backslash \mu^{-1}(\alpha)$ and we have seen in 3.9 that $\widehat{\Phi}(\bar{a}, \bar{b}) = \Phi(a, b)\check{}$. $\square$

We shall now compare the Kähler structure obtained on $\mathcal{P}^3_+(\alpha)$ from the Grassmannian to that introduced by Klyachko [Kl] or Kapovich-Millson ([KM2], §3). Using the standard cross product $\times$ and scalar product $\langle .,. \rangle$ on $\mathbf{R}^3$, these authors put on the sphere S^2_r of radius r the complex structure $\widetilde{J}$

defined by

$$\widetilde{J}v := \frac{1}{r}\, x \times v \qquad (v \in T_x S_r^2)$$

and the Kähler metric

$$\widetilde{h}(u,v) := \frac{1}{r}\langle u, v\rangle - \frac{i}{r^2}\langle x, u \times v\rangle \qquad (u, v \in T_x S_r^2)$$

with associated symplectic form $\widetilde{\omega}(u,v) := \left\langle \frac{x}{r^2} u \times v \right\rangle$. Let $W(\alpha) := \prod_{i=1}^m S^2_{\alpha_i}$. The map $\beta : W_\alpha \longrightarrow \mathbf{R}^3$ defined by $\beta(z_1, \dots, z_m) := \sum_{i=1}^m z_i$ is the moment map for the diagonal action of SO_3 on W_α. The space $\mathcal{P}^3_+(\alpha)$ thus occurs as the symplectic reduction $SO_3 \backslash \beta^{-1}(0)$.

PROPOSITION 4.5. *The complex structure J and Kähler metric h of 4.4 compare with those $\widetilde{J}$ and $\widetilde{h}$ of Kapovich-Millson in the following way:*

$$\widetilde{J} = J \quad \textit{and} \quad \widetilde{h}(u,v) = 4\, h(u,v)\,.$$

Proof. Starting from the Hermitian vector space $\mathcal{M} = \mathcal{M}_{m\times 2}(\mathbf{C})$ one sees that $\mathcal{P}^3(\alpha)$ is obtained by two successive symplectic reductions

$$\mathbf{G}_2(\mathbf{C}^m) = \widetilde{\Phi}^{-1}(0)/U_2 \quad \text{and} \quad \mathcal{P}^3(\alpha) = U_1^m \backslash \mu^{-1}(\alpha)$$

(we use the notation of §3). One can perform the reductions in the reverse order. We first get

$$U_1^m \backslash \widetilde{\Psi}^{-1}(\alpha) = \prod_{i=1}^m \mathbf{C}P^1_{\alpha_i}$$

where $\mathbf{C}P^1_r$ is the quotient of the 3-dimensional sphere

$$\{(u,v) \in \mathbf{C}^2 \mid |u|^2 + |v|^2 = r\}$$

by the diagonal action of U_1. The moment map $\widetilde{\Phi} : \mathcal{M} \longrightarrow \mathcal{H}(2)$ gives a a moment map (still called $\widetilde{\Phi}$) from the product of projective spaces into $\mathcal{H}_0(2)$. One has a commutative diagram

$$\begin{array}{ccc}
\prod_{i=1}^m \mathbf{C}P^1_{\alpha_i} & \xrightarrow[\simeq]{\Pi^\phi} & \prod_{i=1}^m S^2_{\alpha_i} \\
{\scriptstyle \widetilde{\Phi}}\big\downarrow & & \big\downarrow{\scriptstyle \beta} \\
\mathcal{H}_0(2) & \xrightarrow[\simeq]{\psi} & \mathbf{R}^3
\end{array}$$

where $\psi : \mathcal{H}_0(2) \to \mathbf{R}^3 \simeq \mathbf{R} \times \mathbf{C}$ sends the matrix $\begin{pmatrix} u & z \\ \bar{z} & -u \end{pmatrix}$ to (u, z).

To prove Proposition 4.5, it is enough to establish that for all $a \in \mathbf{C}P^1_r$, the tangent map $T_a\phi : T_a\mathbf{C}P^1_r \longrightarrow T_{\phi(a)}S^2_r$ satisfies

$$T_a\phi(Jv) = \widetilde{J}T_a\phi(v) \quad \text{and} \quad \widetilde{\omega}\big(T_a\phi(v), T_a\phi(Jv)\big) = 4\omega(v, Jv).$$

By U_2-equivariance, we can restrict ourselves to $a = [\sqrt{r}, 0]$. The tangent space $T_a\mathbf{C}P^1_r$ is identified with $\{0\} \times \mathbf{C}$ and one can take $v = (0, 1)$ and $Jv = (0, i)$. One has $\phi(a) = (r, 0, 0)$,

$$T_a\phi(v) = (0, 2\sqrt{r}, 0), \quad T_a\phi(Jv) = (0, 0, 2\sqrt{r}) = \widetilde{J}T_a\phi(v)$$

and $\widetilde{\omega}\big(T_a\phi(v), T_a\phi(Jv)\big) = 4$, while $\omega(v, Jv) = 1$. □

REMARKS

(4.6) The results of this section show that the spaces $\mathcal{P}^3_+(\alpha)$ for generic α are the symplectic leaves of the Poisson structure on the regular part of ${}^m\mathcal{P}^3_+$, or ${}^m\mathcal{P}\mathcal{P}^3_+$ given in (3.13) and (3.14).

(4.7) If one works in the pure quaternions $I\mathbf{H}$, the complex structure $\widetilde{J}$ on S^2_r becomes

$$\widetilde{J}(v) = \frac{q\,v}{|q|}, \quad (v \in T_qS^2_r = I\mathbf{H}).$$

The sphere S^2_r is a co-adjoint orbit of $U_1(\mathbf{H})$ and the Hermitian form $\widetilde{w}$ is the Kirillov–Kostant form (see [Gu, Theorem 1.1]).

(4.8) The isomorphism between the symplectic reductions of the Grassmannian $\mathbf{G}_2(\mathbf{C}^m)$ and the product of $\mathbf{C}P^1$'s that underlies our results 3.9, 4.4 and the proof of 4.5 is a symplectic version of the Gel′fand-MacPherson correspondence ([GM] and [GGMS]). The fact that this isomorphism comes from two reductions of $\mathcal{M}$ is the philosophy of "dual pairs" (see [Mo] and the references therein).

5. THE GEL′FAND-CETLIN ACTION

On ${}^m\mathcal{F}^k$ we have so far defined the length functions $\widetilde{\ell}$ measuring the distances between successive vertices. We now introduce $\widetilde{d} : {}^m\mathcal{F}^k \rightarrow \mathbf{R}^m$, $\widetilde{d}(\rho) = (|\rho(1)|, |\rho(1) + \rho(2)|, \ldots, |\sum_{i=1}^m \rho(i)|)$, the lengths of the diagonals connecting the vertices to the origin. (Only $m-3$ of these functions are new, as $\widetilde{d}(\rho)_1 = \widetilde{\ell}(\rho)_1$, $\widetilde{d}(\rho)_{m-1} = \widetilde{\ell}(\rho)_m$, and $\widetilde{d}(\rho)_m = 0$. Hereafter we write only ℓ_i, d_i and the ρ is to be understood.)