Finally, ${}^3\mathcal{P}^2 \simeq \mathbf{C}P^1/\{z \sim \bar{z}\}$ is homeomorphic, via the length-side map ℓ, to the solid triangle

$$ {}^3\mathcal{P}^2 = {}^3\mathcal{P}^3 \xrightarrow[\simeq]{\ell} \{(x_1, x_2, x_3) \in \mathbf{R}^3 \mid x_1 + x_2 + x_3 = 2 \quad \text{and} \quad 0 \le x_i \le 1\} $$

with boundary ${}^3\mathcal{P}^1$.

3. Quaternions, Grassmannians and structures on the full polygon spaces

(3.1) Let $\mathbf{H} = \mathbf{C} \oplus \mathbf{C}j$ be the skew-field of quaternions; the space $I\mathbf{H}$ of pure imaginary quaternions is equipped with the orthonormal basis i, j and $k = ij$, giving rise to an isometry with $\mathbf{R}^3$ which turns the pure imaginary part of the quaternionic multiplication pq into the usual cross product $p \times q$. The space ${}^m\mathcal{F}^3$ is thus identified with ${}^m\mathcal{F}(I\mathbf{H})$ which gives rise to the canonical identifications on the the various moduli spaces (see (2.2)).

Recall that the correspondence

$$ \eta : u + vj \mapsto \begin{pmatrix} u & v \\ -\bar{v} & \bar{u} \end{pmatrix} $$

gives an injective $\mathbf{R}$-algebra homomorphism $\eta : \mathbf{H} \longrightarrow \mathcal{M}_{(2\times 2)}(\mathbf{C})$. This enables a matrix $P \in U_2$ to act on the right or on the left on $\mathbf{H}$. It also identifies the group S^3 of unit quaternions with SU_2.

(3.2) The Hopf map $\phi : \mathbf{H} \longrightarrow I\mathbf{H}$ defined by

$$ \phi(q) := \bar{q}\, i\, q $$

sends the 3-sphere of radius $\sqrt{r}$ in $\mathbf{H}$ onto the 2-sphere of radius r in $I\mathbf{H}$. (The formulae given in the original paper by Hopf [Ho, §5] actually correspond to the map $q \mapsto \bar{q}kq$.) The equality $\phi(q) = \phi(q')$ occurs if and only if $q' = e^{i\theta} q$. The map ϕ satisfies the equivariance relation $\phi(q \cdot P) = P^{-1} \cdot \phi(q) \cdot P$. Writing $q = u + vj$ with $u, v \in \mathbf{C}$, one has

$$ \phi(u + vj) = (\bar{u} - j\bar{v})\, i(u + vj) = i(\bar{u} + j\bar{v})\,(u + vj) = i\big[(|u|^2 - |v|^2) + 2\bar{u}vj\big]\,. $$

(3.3) Observe that if $q = s + tj$ with $s, t \in \mathbf{R}$, then $\phi(q) = i\,q^2$. This plane $\mathbf{R} \oplus \mathbf{R}j$ of its images is the fixed point set of the involution $a + bj \mapsto \bar{a} + \bar{b}j$ that will be used later. Its image under ϕ is $\mathbf{R}i \oplus \mathbf{R}k$.

(3.4) Remark. $I\mathbf{H}$, with the Lie bracket $[p, q] = pq - qp = 2\,\mathrm{Im}(pq)$, is the Lie algebra for the group $U_1(\mathbf{H}) \simeq SU_2 \simeq S^3$. The pairing

$(q,q') \mapsto -\mathrm{Re}(qq') = \langle q,q'\rangle$ identifies $I\mathbf{H}$ with its dual. If $\mathbf{H} \simeq \mathbf{C}\oplus\mathbf{C}$ is endowed with the standard Kähler form, then the map $\frac{1}{2}\phi$ is the moment map for the Hamiltonian action of $U_1(\mathbf{H})$ on $\mathbf{H}$ (the factor $\frac{1}{2}$ can be checked by restricting the action to the S^1-action on $\mathbf{C}$).

(3.5) Let $\mathbf{V}_2(\mathbf{C}^m)$ be the space of $(m\times 2)$-matrices

$$(a,b) := \begin{pmatrix} a_1 & b_1 \\ \vdots & \vdots \\ a_m & b_m \end{pmatrix} \in \mathcal{M}_{m\times 2}(\mathbf{C})$$

such that $|a| = |b| = 1$ and $\langle a,b\rangle = 0$. $\mathbf{V}_2(\mathbf{C}^m)$ is the Stiefel manifold of orthonormal 2-frames in $\mathbf{C}^m$. The group U_m acts transitively on the left on $\mathbf{V}_2(\mathbf{C}^m)$ producing the diffeomorphism $\mathbf{V}_2(\mathbf{C}^m) = U_m/U_{m-2}$. One has the conjugation on $\mathbf{V}_2(\mathbf{C}^m)$ given by $(a,b)\mapsto(\bar a,\bar b)$ with fixed-point space the Stiefel manifold $\mathbf{V}_2(\mathbf{R}^m) = O_m/O_{m-2}$ of orthonormal 2-frames in $\mathbf{R}^m$. Finally, the embedding $\mathbf{V}_2(\mathbf{C}^m)\subset\mathbf{H}^m$ given by $(a,b)\mapsto(\dots,a_r+b_rj,\dots)$ intertwines the conjugation on $\mathbf{V}_2(\mathbf{C}^m)$ with the involution of (2.5) on $\mathbf{H}^m$. One thus gets an embedding $\mathbf{V}_2(\mathbf{R}^m)\subset(\mathbf{R}\oplus\mathbf{R}j)^m$.

Using the Hopf map ϕ of (3.2), one defines the smooth map $\Phi:\mathbf{V}_2(\mathbf{C}^m)\longrightarrow {}^m\mathcal{F}(I\mathbf{H})\simeq {}^m\mathcal{F}^3$ by the formula

$$\Phi(a,b) := \big(\phi(a_1+b_1j),\phi(a_2+b_2j),\dots,\phi(a_m+b_mj)\big)\,.$$

The fact that $\sum\phi(a_r+b_rj)=0$ is equivalent to $\langle a,b\rangle = 0$ and $|a|=|b|$. As $|a|=|b|=1$, the image of Φ is exactly $S({}^m\mathcal{F}^3)$. By composing with the projection ${}^m\mathcal{F}^3-\{0\}\longrightarrow{}^m\widetilde{\mathcal{P}}^3$, one gets a surjective smooth map $\Phi\colon\mathbf{V}_2(\mathbf{C}^m)\longrightarrow{}^m\widetilde{\mathcal{P}}^3$. One checks that $\Phi(a,b)=\Phi(a',b')$ if and only if (a,b) and (a',b') are in the same orbit under the action of the maximal torus U_1^m of diagonal matrices in U_m. This action is free when none of the (a_i,b_i)'s vanishes, namely if and only if $\Phi(a,b)$ is a proper polygon. As $\Phi(\bar a,\bar b)=\Phi(a,b)^{\vee}$, the restriction of Φ to the fixed points gives a smooth map $\Phi_{\mathbf{R}}:\mathbf{V}_2(\mathbf{R}^m)\longrightarrow{}^m\widetilde{\mathcal{P}}(\mathbf{R}i\oplus\mathbf{R}k)\simeq{}^m\widetilde{\mathcal{P}}^2$ with analogous properties. We have thus proved

THEOREM 3.6. *a) The smooth map* $\Phi:\mathbf{V}_2(\mathbf{C}^m)\longrightarrow{}^m\widetilde{\mathcal{P}}^3$ *induces a homeomorphism* $\widehat{\Phi}:U_1^m\backslash\mathbf{V}_2(\mathbf{C}^m)\xrightarrow{\simeq}{}^m\widetilde{\mathcal{P}}^3$ *such that* $\widehat{\Phi}(\bar a,\bar b)=\Phi(a,b)^{\vee}$. *The restriction of* Φ *above the space of proper polygons is a smooth principal* U_1^m*-bundle.*

b) The smooth map $\Phi_{\mathbf{R}}:\mathbf{V}_2(\mathbf{R}^m)\longrightarrow{}^m\widetilde{\mathcal{P}}^2$ *induces a homeomorphism* $\widehat{\Phi}_{\mathbf{R}}:O_1^m\backslash\mathbf{V}_2(\mathbf{R}^m)\xrightarrow{\simeq}{}^m\widetilde{\mathcal{P}}^2$. *The restriction of* $\Phi_{\mathbf{R}}$ *above the space of proper planar polygons is a principal* O_1^m*-covering.*

COROLLARY 3.7. *${}^m\widetilde{\mathcal{P}}^3 \simeq U_1^m\backslash U_m/U_{m-2}$ and ${}^m\widetilde{\mathcal{P}}^2 \simeq O_1^m\backslash O_m/O_{m-2}$.*

(3.8) Let $\mathbf{G}_2(\mathbf{C}^m)$ be the Grassmann manifold of 2-planes in $\mathbf{C}^m$. The map $\mathbf{V}_2(\mathbf{C}^m) \longrightarrow \mathbf{G}_2(\mathbf{C}^m)$ which associates to (a,b) the plane generated by a and b is the projection $\mathbf{V}_2(\mathbf{C}^m) \longrightarrow \mathbf{V}_2(\mathbf{C}^m)/U_2$ (a principal U_2 bundle), for the natural right action of U_2 on $\mathbf{V}_2(\mathbf{C}^m) \subset \mathcal{M}_{m\times 2}(\mathbf{C})$. This projection is U_m-equivariant, equivalent to the projection $U_m/U_{m-2} \longrightarrow U_m/U_2 \times U_{m-2}$.

The map $\Phi : \mathbf{V}_2(\mathbf{C}^m) \longrightarrow {}^m\widetilde{\mathcal{P}}^3$ satisfies

$$\Phi\big((a,b)P\big) = P^{-1}\,\Phi(a,b)\,P \quad \text{for} \quad (a,b) \in \mathbf{V}_2(\mathbf{C}^m),\ P \in U_2\ .$$

The conjugation by P being an element of $SO(I\mathbf{H})$, one thus gets a map (still called Φ) from $\mathbf{G}_2(\mathbf{C}^m)$ onto ${}^m\mathcal{P}^3_+$. The space ${}^m\mathcal{P}^3_+$ has a smooth structure on the open-dense subset of non-lined polygons (which is where the SO_3-action was free) and, above this open-dense subset, the new map Φ is smooth. The map Φ intertwines the involutions and so restricts to a map $\Phi_{\mathbf{R}}\colon \mathbf{G}_2(\mathbf{R}^m) \longrightarrow {}^m\mathcal{P}^2$, where $\mathbf{G}_2(\mathbf{R}^m)$ is the Grassmannian of 2-planes in $\mathbf{R}^m$. In this case, an intermediate object is the Grassmannian $\widetilde{\mathbf{G}}_2(\mathbf{R}^m) = SO_m/SO_2 \times SO_{m-2}$ of oriented 2-planes in $\mathbf{R}^m$ with the smooth map $\Phi_{\mathbf{R}}\widetilde{\mathbf{G}}_2(\mathbf{R}^m) \longrightarrow {}^m\mathcal{P}^2_+ \simeq \mathbf{C}P^{m-2}$. The action of U_1^m on $\mathbf{V}_2(\mathbf{C}^m)$ descends to an action on $\mathbf{G}_2(\mathbf{C}^m)$ which is no longer effective: its kernel is the diagonal subgroup Δ of U_1^m, the center of U_m, isomorphic to U_1. The same holds true in the real case, replacing U_1 by O_1 (the diagonal subgroup of O_1^m is also denoted by Δ).

Using Theorem 3.6, the reader will easily prove the following

THEOREM 3.9. *a) The map $\Phi : \mathbf{G}_2(\mathbf{C}^m) \longrightarrow {}^m\mathcal{P}^3$ induces a homeomorphism $\widehat{\Phi} : U_1^m\backslash\mathbf{G}_2(\mathbf{C}^m) \xrightarrow{\simeq} {}^m\mathcal{P}^3$ such that $\widehat{\Phi}(\bar{a},\bar{b}) = \Phi(a,b)^\vee$. The restriction of $\widehat{\Phi}$ above the space of proper non-lined polygons is a smooth principal (U_1^m/Δ)-bundle.*

b) The smooth map $\Phi_{\mathbf{R}} : \widetilde{\mathbf{G}}_2(\mathbf{R}^m) \longrightarrow {}^m\mathcal{P}^2_+$ induces a homeomorphism $\widehat{\Phi}_{\mathbf{R}} : O_1^m\backslash\widetilde{\mathbf{G}}_2(\mathbf{R}^m) \xrightarrow{\simeq} {}^m\mathcal{P}^2_+$. It is a smooth branched covering and, restricted above the space of proper polygons, a principal (O_1^m/Δ)-covering.

c) The map $\Phi_{\mathbf{R}} : \mathbf{G}_2(\mathbf{R}^m) \longrightarrow {}^m\mathcal{P}^2$ induces a homeomorphism $\widehat{\Phi}_{\mathbf{R}} : O_1^m\backslash\mathbf{G}_2(\mathbf{R}^m) \xrightarrow{\simeq} {}^m\mathcal{P}^2$. The restriction of $\widehat{\Phi}$ above the space of proper non-lined polygons is a principal (O_1^m/Δ)-covering.

COROLLARY 3.10. *One has homeomorphisms between the polygon spaces and the double cosets*

a) ${}^m\mathcal{P}^3 \simeq U_1^m \backslash U_m/(U_2 \times U_{m-2})$

b) ${}^m\mathcal{P}^2_+ \simeq S(O_1^m)\backslash SO_m/(SO_2 \times SO_{m-2})$.

c) ${}^m\mathcal{P}^2 \simeq O_1^m \backslash O_m/(O_2 \times O_{m-2})$.

(3.11) *Example.* As in (2.7) the example of planar triangles ($m = 3$ and $k = 2$) is interesting. The Stiefel manifold $\mathbf{V}_2(\mathbf{R}^3)$ is diffeomorphic to the unit tangent bundle to S^2, in turn diffeomorphic to SO_3. The oriented Grassmannian $\widetilde{\mathbf{G}}_2(\mathbf{R}^3)$ can be identified with S^2 by associating to an oriented plane its unit normal vector. The smooth map

$$\Phi_{\mathbf{R}} : S^2 \simeq \widetilde{\mathbf{G}}_2(\mathbf{R}^3)) \longrightarrow {}^3\mathcal{P}^2_+ \simeq S^2$$

is of degree 4, branched over the 3 points. This map can be visualized as follows: tesselate $\mathbf{R}^2$ with equilateral triangles. Divide $\mathbf{R}^2$ by the subgroup of isometries which preserve the tesselation and the orientation (it thus preserves a checkerboard coloring of the triangle tesselation). This quotient is a well known orbifold structure on S^2 with three branched points. The projection $\mathbf{R}^2 \longrightarrow S^2$ factors through an octahedron with a chess-board coloring of its faces. The residual map from this octahedron to S^2 is our map $\Phi_{\mathbf{R}}$.

Take the pullback by $\Phi_{\mathbf{R}}$ of the Hopf bundle $S^3 \longrightarrow S^2$. One gets a map of degree 4 from some lens space L onto S^3, with branched locus the link formed by three SO_2-orbits. The lens space will be doubly covered by SO_3. We thus get the map

$$\widetilde{\Phi} : SO_3 \simeq \mathbf{V}_2(\mathbf{R}^3) \longrightarrow {}^3\widetilde{\mathcal{P}}^2 \simeq S^3$$

of degree 8. Finally, one has $\mathbf{G}_2(\mathbf{R}^3) \simeq \mathbf{R}P^2$ and $\Phi_{\mathbf{R}}$ is the quotient of $\mathbf{R}P^2$ by the action of O_1^3 on each homogeneous coordinate. This quotient is a 2-simplex and one sees again that ${}^3\mathcal{P}^2$ is a solid triangle.

(3.12) *Orbifold structures.* The maps $\widehat{\Phi}_{\mathbf{R}}$ and $\Phi_{\mathbf{R}}$ provide, for the spaces ${}^2\widetilde{\mathcal{P}}^2 \simeq S^{2m-3}$ and ${}^m\mathcal{P}^2_+ \simeq \mathbf{C}P^{m-2}$, a smooth orbifold structure. Each point has a neighbourhood homeomorphic to an open set of the quotient of $(\mathbf{R}^2)^s$ by a subgroup of O_1^s, where O_1 acts on each $\mathbf{R}^2$ via the antipodal map. Observe that the map $\Phi_{\mathbf{R}}$ is a "small cover" in the sense of [DJ]. The branched loci are $E_{m-1}{}^m\widetilde{\mathcal{P}}^2$ and $E_{m-1}{}^m\mathcal{P}^2_+$ respectively. As for ${}^m\mathcal{P}^2$ we have to add the branched locus ${}^m\mathcal{P}^1$. The generic points of ${}^m\mathcal{P}^1$ have a neighbourhood modelled on the quotient of $\mathbf{C}^{m-2}$ by complex conjugation.

Analogously, the map $\Phi\colon \mathbf{G}_2(\mathbf{C}^m) \longrightarrow {}^m\mathcal{P}^3$ gives rise, for the space ${}^m\widetilde{\mathcal{P}}^3$, to a smooth *complex orbifold structure*. By that we mean a space locally modelled on the quotient of $\mathbf{C}^s$ by a subgroup of U_1^s. We define the space $\mathcal{C}^\infty({}^m\mathcal{P}^3)$ of *smooth* maps from ${}^m\mathcal{P}^3$ to the reals as the subspace of $\mathcal{C}^\infty\big(\mathbf{G}_2(\mathbf{C}^m)\big)$ which is invariant by the action of U_1^m.

(3.13) *Riemannian and Poisson structures.* Let $\mathcal{H}(m)$ be the space of Hermitian $(m \times m)$-matrices, identified with $\mathbf{u}_m^*$ via the pairing

$$\mathcal{H}(m) \times \mathbf{u_m} \longrightarrow \mathbf{R} \qquad (H, X) \mapsto \frac{i}{2}\,\mathrm{tr}(HX)\,.$$

This identification turns the co-adjoint action of U_m into the conjugation action on $\mathcal{H}(m)$. Consider the map $\widetilde{\Psi} : \mathcal{M}_{m\times 2}(\mathbf{C}) \longrightarrow \mathcal{H}(m)$ given by $\widetilde{\Psi}(a,b) := (a,b)\cdot(a,b)^*$. One has $\widetilde{\Psi}\big(Q\cdot(a,b)\cdot P\big) = Q\cdot\widetilde{\Psi}((a,b)\cdot Q^*$ for $P \in U_2$ and $Q \in U_m$ and thus $\mathcal{C} := \widetilde{\Psi}\big(\mathbf{V}_2(\mathbf{C}^m)\big)$ is the U_m-orbit through $\mathrm{diag}(1,1,0,\ldots,0)$. This proves that $\widetilde{\Psi}$ descends to a diffeomorphism $\Psi : \mathbf{G}_2(\mathbf{C}^m) \xrightarrow{\simeq} \mathcal{C}$.

The complex vector space $\mathcal{M}_{m\times 2}(\mathbf{C})$ is endowed with its classical Hermitian structure $\langle A, B\rangle := \mathrm{tr}(AB^*)$, with associated symplectic form $\omega(\ ,\ ) = -\,\mathrm{Im}\,\langle\ ,\ \rangle$. The map $\widetilde{\Psi}$ above and the map $\widetilde{\Phi} : \mathcal{M}_{m\times 2}(\mathbf{C}) \longrightarrow \mathcal{H}_0(2)$ given by

$$\widetilde{\Phi}(a,b) := (a,b)^*\cdot(a,b) - \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

are moment maps for the Hamiltonian actions of U_m and U_2 respectively. One has $\mathbf{V}_2(\mathbf{C}^m) = \widetilde{\Phi}^{-1}(0)$ and thus $\mathbf{G}_2(\mathbf{C}^m)$ occurs as symplectic reduction of the Hermitian vector space $\mathcal{M}_{m\times 2}(\mathbf{C})$ and thereby inherits a U_m-invariant Kähler structure, using, for instance [Ki], § 1.7. (Strictly speaking, one deals in [Ki] with compact Kähler manifolds; to fulfill this condition, one can first divide $\mathcal{M}_{m\times 2}(\mathbf{C}) - \{0\}$ by the diagonal action of $\mathbf{C}^*$ to put oneself into a complex projective space.) The residual map $\Psi : \mathbf{G}_2(\mathbf{C}^m) \xrightarrow{\simeq} \mathcal{C} \subset \mathcal{H}(m)$ is a moment map for the action of U_m on $\mathbf{G}_2(\mathbf{C}^m)$.

Being thus a Kähler manifold, $\mathbf{G}_2(\mathbf{C}^m)$ is a Riemannian Poisson manifold. This structure descends to the complex orbifold ${}^m\mathcal{P}^3$: the algebra $\mathcal{C}^\infty({}^m\mathcal{P}^3)$ admits a unique Lie bracket so that the projection $\mathbf{G}_2(\mathbf{C}^m) \longrightarrow {}^m\mathcal{P}^3$ is a Poisson map.

(3.14) It is possible to endow with a Poisson structure the space ${}^m\mathcal{PP}^3_+$ of configurations of *all* m-gons in $\mathbf{R}^3$, without fixing the perimeter to 2. It suffices in the above construction, to replace the U_2-reduction $\mathbf{G}_2(\mathbf{C}^m) = \widetilde{\Phi}^{-1}(0)/U_2$ by the SU_2-reduction $\widetilde{\mathbf{G}}_2(\mathbf{C}^m) := \widetilde{\Phi}^{-1}(0)/SU_2$. The latter is a non-compact space, the total space of the determinant bundle over $\mathbf{G}_2(\mathbf{C}^m)$ with the zero section collapsed. The trace function on $\mathcal{M}_{m\times 2}(\mathbf{C})$ descends to $\widetilde{\mathbf{G}}_2(\mathbf{C}^m)$ and to the Casimir function "perimeter" on ${}^m\mathcal{P}\mathcal{P}^3_+$.

4. Polygons with given sides – Kähler structures

We now use the map $\ell : {}^m\widetilde{\mathcal{P}}^k, {}^m\mathcal{P}^k_+, {}^m\mathcal{P}^k \to \mathbf{R}^m$ defined in (2.4). Recall that $\ell(\rho)$, for $\rho \in {}^m\widetilde{\mathcal{P}}^k$, is the length of the successive sides of a representative of r with total perimeter 2.

For $\alpha = (\alpha_1, \dots, \alpha_m) \in \mathbf{R}^m_{\geq 0}$ with $\sum_{i=1}^m \alpha_i = 2$, we define

$$ {}^m\widetilde{\mathcal{P}}^k(\alpha) =: \widetilde{\mathcal{P}}^k(\alpha) := \{\rho \in {}^m\widetilde{\mathcal{P}}^k \mid \ell(\rho) = \alpha\} \subset {}^m\widetilde{\mathcal{P}}^k . $$

The space $\widetilde{\mathcal{P}}^k(\alpha)$ is invariant under the action of O_k. We define the moduli spaces

$$ \mathcal{P}^k_+(\alpha) := SO_k \backslash \widetilde{\mathcal{P}}^k(\alpha) = \ell^{-1}(\alpha) \subset {}^m\mathcal{P}^k_+ $$

and

$$ \mathcal{P}^k(\alpha) := O_k \backslash \widetilde{\mathcal{P}}^k(\alpha) = \ell^{-1}(\alpha) \subset {}^m\mathcal{P}^k . $$

The space $\widetilde{\mathcal{P}}^1(\alpha)$ consists of a finite number of points and is generically empty. We call α *generic* if $\widetilde{\mathcal{P}}^1(\alpha) = \varnothing$.

THEOREM 4.1. *The map $\mu := \ell \circ \widehat{\Phi} : \mathbf{G}_2(\mathbf{C}^m) \longrightarrow \mathbf{R}^m$ is a moment map for the action of U_1^m on $\mathbf{G}_2(\mathbf{C}^m)$.*

Proof. As seen in (3.13), the moment map $\Psi : \mathbf{G}_2(\mathbf{C}^m) \longrightarrow \mathcal{H}(m)$ for the U_m-action on $\mathbf{G}_2(\mathbf{C}^m)$ is induced from $\widetilde{\Psi} : \mathcal{M}_{m\times 2}(\mathbf{C}) \longrightarrow \mathcal{H}(m)$ given by $\widetilde{\Psi}(a, b) := (a, b) \cdot (a, b)^*$. A moment map μ for the action of U_1^m is obtained by composing Ψ with the projection $\mathcal{H}(m) \longrightarrow \mathbf{R}^m$ associating to a matrix its diagonal entries. So, if $\Pi \in \mathbf{G}_2(\mathbf{C}^m)$ is generated by a and b with $(a, b) \in \mathbf{V}_2(\mathbf{C}^m)$, one has

$$ \mu(\Pi) = (|a_1|^2 + |b_1|^2, \dots, |a_m|^2 + |b_m|^2) = \ell \circ \widehat{\Phi}(a, b) . \qquad \square $$

A now classic theorem of Atiyah and Guillemin-Sternberg [Au, § III.4.2] asserts that the image of a moment map for a torus action is a convex polytope (the *moment polytope*). The restriction of the moment map to the fixed point set of an anti-symplectic involution has the same image [Du]. In our case, one gets these facts directly: