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Finally, ${}^3\mathcal{P}^2 \simeq \mathbb{C}P^1/\{z \sim \overline{z}\}$ is homeomorphic, via the length-side map ℓ , to the solid triangle

$${}^{3}\mathcal{P}^{2} = {}^{3}\mathcal{P}^{3} \xrightarrow{\ell} \{(x_{1}, x_{2}, x_{3}) \in \mathbf{R}^{3} \mid x_{1} + x_{2} + x_{3} = 2 \text{ and } 0 \le x_{i} \le 1\}$$
 with boundary ${}^{3}\mathcal{P}^{1}$.

3. QUATERNIONS, GRASSMANNIANS AND STRUCTURES ON THE FULL POLYGON SPACES

(3.1) Let $\mathbf{H} = \mathbf{C} \oplus \mathbf{C} j$ be the skew-field of quaternions; the space $I\mathbf{H}$ of pure imaginary quaternions is equipped with the orthonormal basis i, j and k = ij, giving rise to an isometry with \mathbf{R}^3 which turns the pure imaginary part of the quaternionic multiplication pq into the usual cross product $p \times q$. The space ${}^m\mathcal{F}^3$ is thus identified with ${}^m\mathcal{F}(I\mathbf{H})$ which gives rise to the canonical identifications on the the various moduli spaces (see (2.2)).

Recall that the correspondence

$$\eta: u + vj \mapsto \begin{pmatrix} u & v \\ -\overline{v} & \overline{u} \end{pmatrix}$$

gives an injective **R**-algebra homomorphism $\eta: \mathbf{H} \longrightarrow \mathcal{M}_{(2\times 2)}(\mathbf{C})$. This enables a matrix $P \in U_2$ to act on the right or on the left on **H**. It also identifies the group S^3 of unit quaternions with SU_2 .

(3.2) The Hopf map $\phi: \mathbf{H} \longrightarrow I\mathbf{H}$ defined by

$$\phi(q) := \overline{q} i q$$

sends the 3-sphere of radius \sqrt{r} in **H** onto the 2-sphere of radius r in I**H**. (The formulae given in the original paper by Hopf [Ho, §5] actually correspond to the map $q \mapsto \overline{q}kq$.) The equality $\phi(q) = \phi(q')$ occurs if and only if $q' = e^{i\theta} q$. The map ϕ satisfies the equivariance relation $\phi(q \cdot P) = P^{-1} \cdot \phi(q) \cdot P$. Writing q = u + vj with $u, v \in \mathbb{C}$, one has

$$\phi(u+vj) = (\overline{u}-j\overline{v})i(u+vj) = i(\overline{u}+j\overline{v})(u+vj) = i[(|u|^2-|v|^2)+2\overline{u}vj].$$

- (3.3) Observe that if q = s + tj with $s, t \in \mathbf{R}$, then $\phi(q) = i q^2$. This plane $\mathbf{R} \oplus \mathbf{R} j$ of its images is the fixed point set of the involution $a + bj \mapsto \overline{a} + \overline{b}j$ that will be used later. Its image under ϕ is $\mathbf{R} i \oplus \mathbf{R} k$.
- (3.4) REMARK. $I\mathbf{H}$, with the Lie bracket $[p,q]=pq-qp=2\operatorname{Im}(pq)$, is the Lie algebra for the group $U_1(\mathbf{H})\simeq SU_2\simeq S^3$. The pairing

 $(q, q') \mapsto -\operatorname{Re}(qq') = \langle q, q' \rangle$ identifies $I\mathbf{H}$ with its dual. If $\mathbf{H} \simeq \mathbf{C} \oplus \mathbf{C}$ is endowed with the standard Kähler form, then the map $\frac{1}{2}\phi$ is the moment map for the Hamiltonian action of $U_1(\mathbf{H})$ on \mathbf{H} (the factor $\frac{1}{2}$ can be checked by restricting the action to the S^1 -action on \mathbf{C}).

(3.5) Let $V_2(\mathbb{C}^m)$ be the space of $(m \times 2)$ -matrices

$$(a,b) := \begin{pmatrix} a_1 & b_1 \\ \vdots & \vdots \\ a_m & b_m \end{pmatrix} \in \mathcal{M}_{m \times 2}(\mathbf{C})$$

such that |a| = |b| = 1 and $\langle a, b \rangle = 0$. $\mathbf{V}_2(\mathbf{C}^m)$ is the Stiefel manifold of orthonormal 2-frames in \mathbf{C}^m . The group U_m acts transitively on the left on $\mathbf{V}_2(\mathbf{C}^m)$ producing the diffeomorphism $\mathbf{V}_2(\mathbf{C}^m) = U_m/U_{m-2}$. One has the conjugation on $\mathbf{V}_2(\mathbf{C}^m)$ given by $(a,b) \mapsto (\overline{a},\overline{b})$ with fixed-point space the Stiefel manifold $\mathbf{V}_2(\mathbf{R}^m) = O_m/O_{m-2}$ of orthonormal 2-frames in \mathbf{R}^m . Finally, the embedding $\mathbf{V}_2(\mathbf{C}^m) \subset \mathbf{H}^m$ given by $(a,b) \mapsto (\ldots,a_r+b_rj,\ldots)$ intertwines the conjugation on $\mathbf{V}_2(\mathbf{C}^m)$ with the involution of (2.5) on \mathbf{H}^m . One thus gets an embedding $\mathbf{V}_2(\mathbf{R}^m) \subset (\mathbf{R} \oplus \mathbf{R}j)^m$.

Using the Hopf map ϕ of (3.2), one defines the smooth map $\Phi: \mathbf{V}_2(\mathbf{C}^m) \longrightarrow {}^m \mathcal{F}(I\mathbf{H}) \simeq {}^m \mathcal{F}^3$ by the formula

$$\Phi(a,b) := (\phi(a_1 + b_1 j), \phi(a_2 + b_2 j), \dots, \phi(a_m + b_m j)).$$

The fact that $\sum \phi(a_r + b_r j) = 0$ is equivalent to $\langle a, b \rangle = 0$ and |a| = |b|. As |a| = |b| = 1, the image of Φ is exactly $S(^m \mathcal{F}^3)$. By composing with the projection $^m \mathcal{F}^3 - \{0\} \longrightarrow ^m \widetilde{\mathcal{P}}^3$, one gets a surjective smooth map $\Phi \colon \mathbf{V}_2(\mathbf{C}^m) \longrightarrow ^m \widetilde{\mathcal{P}}^3$. One checks that $\Phi(a,b) = \Phi(a',b')$ if and only if (a,b) and (a',b') are in the same orbit under the action of the maximal torus U_1^m of diagonal matrices in U_m . This action is free when none of the (a_i,b_i) 's vanishes, namely if and only if $\Phi(a,b)$ is a proper polygon. As $\Phi(\overline{a},\overline{b}) = \Phi(a,b)$, the restriction of Φ to the fixed points gives a smooth map $\Phi_{\mathbf{R}}: \mathbf{V}_2(\mathbf{R}^m) \longrightarrow ^m \widetilde{\mathcal{P}}(\mathbf{R}\,i \oplus \mathbf{R}\,k) \simeq ^m \widetilde{\mathcal{P}}^2$ with analogous properties. We have thus proved

THEOREM 3.6. a) The smooth map $\Phi: \mathbf{V}_2(\mathbf{C}^m) \longrightarrow {}^m\widetilde{\mathcal{P}}^3$ induces a homeomorphism $\widehat{\Phi}: U_1^m \backslash \mathbf{V}_2(\mathbf{C}^m) \stackrel{\simeq}{\longrightarrow} {}^m\widetilde{\mathcal{P}}^3$ such that $\widehat{\Phi}(\overline{a}, \overline{b}) = \Phi(a, b)$. The restriction of Φ above the space of proper polygons is a smooth principal U_1^m -bundle.

b) The smooth map $\Phi_{\mathbf{R}}: \mathbf{V}_2(\mathbf{R}^m) \longrightarrow {}^m\widetilde{\mathcal{P}}^2$ induces a homeomorphism $\widehat{\Phi}_{\mathbf{R}}: O_1^m \backslash \mathbf{V}_2(\mathbf{R}^m) \xrightarrow{\cong} {}^m\widetilde{\mathcal{P}}^2$. The restriction of $\Phi_{\mathbf{R}}$ above the space of proper planar polygons is a principal O_1^m -covering.

COROLLARY 3.7.
$$^{m}\widetilde{\mathcal{P}}^{3} \simeq U_{1}^{m} \backslash U_{m}/U_{m-2}$$
 and $^{m}\widetilde{\mathcal{P}}^{2} \simeq O_{1}^{m} \backslash O_{m}/O_{m-2}$.

(3.8) Let $G_2(\mathbb{C}^m)$ be the Grassmann manifold of 2-planes in \mathbb{C}^m . The map $V_2(\mathbb{C}^m) \longrightarrow G_2(\mathbb{C}^m)$ which associates to (a,b) the plane generated by a and b is the projection $V_2(\mathbb{C}^m) \longrightarrow V_2(\mathbb{C}^m)/U_2$ (a principal U_2 bundle), for the natural right action of U_2 on $V_2(\mathbb{C}^m) \subset \mathcal{M}_{m \times 2}(\mathbb{C})$. This projection is U_m -equivariant, equivalent to the projection $U_m/U_{m-2} \longrightarrow U_m/U_2 \times U_{m-2}$.

The map $\Phi: \mathbf{V}_2(\mathbf{C}^m) \longrightarrow {}^m\widetilde{\mathcal{P}}^3$ satisfies

$$\Phi((a,b)P) = P^{-1}\Phi(a,b)P$$
 for $(a,b) \in \mathbf{V}_2(\mathbf{C}^m), P \in U_2$.

The conjugation by P being an element of $SO(I\mathbf{H})$, one thus gets a map (still called Φ) from $\mathbf{G}_2(\mathbf{C}^m)$ onto ${}^m\mathcal{P}_+^3$. The space ${}^m\mathcal{P}_+^3$ has a smooth structure on the open-dense subset of non-lined polygons (which is where the SO_3 -action was free) and, above this open-dense subset, the new map Φ is smooth. The map Φ intertwines the involutions and so restricts to a map $\Phi_{\mathbf{R}} \colon \mathbf{G}_2(\mathbf{R}^m) \longrightarrow {}^m\mathcal{P}^2$, where $\mathbf{G}_2(\mathbf{R}^m)$ is the Grassmannian of 2-planes in \mathbf{R}^m . In this case, an intermediate object is the Grassmannian $\widetilde{\mathbf{G}}_2(\mathbf{R}^m) = SO_m/SO_2 \times SO_{m-2}$ of oriented 2-planes in \mathbf{R}^m with the smooth map $\Phi_{\mathbf{R}}\widetilde{\mathbf{G}}_2(\mathbf{R}^m) \longrightarrow {}^m\mathcal{P}_+^2 \cong \mathbf{C}P^{m-2}$. The action of U_1^m on $\mathbf{V}_2(\mathbf{C}^m)$ descends to an action on $\mathbf{G}_2(\mathbf{C}^m)$ which is no longer effective: its kernel is the diagonal subgroup Δ of U_1^m , the center of U_m , isomorphic to U_1 . The same holds true in the real case, replacing U_1 by O_1 (the diagonal subgroup of O_1^m is also denoted by Δ).

Using Theorem 3.6, the reader will easily prove the following

THEOREM 3.9. a) The map $\Phi: \mathbf{G}_2(\mathbf{C}^m) \longrightarrow {}^m \mathcal{P}^3$ induces a homeomorphism $\widehat{\Phi}: U_1^m \backslash \mathbf{G}_2(\mathbf{C}^m) \stackrel{\simeq}{\longrightarrow} {}^m \mathcal{P}^3$ such that $\widehat{\Phi}(\overline{a}, \overline{b}) = \Phi(a, b)$. The restriction of $\widehat{\Phi}$ above the space of proper non-lined polygons is a smooth principal (U_1^m/Δ) -bundle.

- b) The smooth map $\Phi_{\mathbf{R}}: \widetilde{\mathbf{G}}_2(\mathbf{R}^m) \longrightarrow {}^m \mathcal{P}_+^2$ induces a homeomorphism $\widehat{\Phi}_{\mathbf{R}}: O_1^m \backslash \widetilde{\mathbf{G}}_2(\mathbf{R}^m) \stackrel{\simeq}{\longrightarrow} {}^m \mathcal{P}_+^2$. It is a smooth branched covering and, restricted above the space of proper polygons, a principal (O_1^m/Δ) -covering.
- c) The map $\Phi_{\mathbf{R}}: \mathbf{G}_2(\mathbf{R}^m) \longrightarrow {}^m\mathcal{P}^2$ induces a homeomorphism $\widehat{\Phi}_{\mathbf{R}}: O_1^m \backslash \mathbf{G}_2(\mathbf{R}^m) \stackrel{\simeq}{\longrightarrow} {}^m\mathcal{P}^2$. The restriction of $\widehat{\Phi}$ above the space of proper non-lined polygons is a principal (O_1^m/Δ) -covering.

COROLLARY 3.10. One has homeomorphisms between the polygon spaces and the double cosets

- a) ${}^m\mathcal{P}^3 \simeq U_1^m \backslash U_m / (U_2 \times U_{m-2})$
- b) ${}^m\mathcal{P}^2_+ \simeq S(O_1^m) \backslash SO_m / (SO_2 \times SO_{m-2}).$
- c) ${}^m\mathcal{P}^2 \simeq O_1^m \backslash O_m / (O_2 \times O_{m-2}).$

(3.11) Example. As in (2.7) the example of planar triangles (m=3 and k=2) is interesting. The Stiefel manifold $\mathbf{V}_2(\mathbf{R}^3)$ is diffeomorphic to the unit tangent bundle to S^2 , in turn diffeomorphic to SO_3 . The oriented Grassmannian $\widetilde{\mathbf{G}}_2(\mathbf{R}^3)$ can be identified with S^2 by associating to an oriented plane its unit normal vector. The smooth map

$$\Phi_{\mathbf{R}}: S^2 \simeq \widetilde{\mathbf{G}}_2(\mathbf{R}^3)) \longrightarrow {}^3\mathcal{P}_+^2 \simeq S^2$$

is of degree 4, branched over the 3 points. This map can be visualized as follows: tesselate \mathbf{R}^2 with equilateral triangles. Divide \mathbf{R}^2 by the subgroup of isometries which preserve the tesselation and the orientation (it thus preserves a checkerboard coloring of the triangle tesselation). This quotient is a well known orbifold structure on S^2 with three branched points. The projection $\mathbf{R}^2 \longrightarrow S^2$ factors through an octahedron with a chess-board coloring of its faces. The residual map from this octahedron to S^2 is our map $\Phi_{\mathbf{R}}$.

Take the pullback by $\Phi_{\mathbf{R}}$ of the Hopf bundle $S^3 \longrightarrow S^2$. One gets a map of degree 4 from some lens space L onto S^3 , with branched locus the link formed by three SO_2 -orbits. The lens space will be doubly covered by SO_3 . We thus get the map

$$\widetilde{\Phi}: SO_3 \simeq \mathbf{V}_2(\mathbf{R}^3) \longrightarrow {}^3\widetilde{\mathcal{P}}^2 \simeq S^3$$

of degree 8. Finally, one has $G_2(\mathbf{R}^3) \simeq \mathbf{R}P^2$ and $\Phi_{\mathbf{R}}$ is the quotient of $\mathbf{R}P^2$ by the action of O_1^3 on each homogeneous coordinate. This quotient is a 2-simplex and one sees again that ${}^3\mathcal{P}^2$ is a solid triangle.

(3.12) Orbifold structures. The maps $\widehat{\Phi}_{\mathbf{R}}$ and $\Phi_{\mathbf{R}}$ provide, for the spaces ${}^2\widetilde{\mathcal{P}}^2\simeq S^{2m-3}$ and ${}^m\mathcal{P}_+^2\simeq \mathbf{C}P^{m-2}$, a smooth orbifold structure. Each point has a neighbourhood homeomorphic to an open set of the quotient of $(\mathbf{R}^2)^s$ by a subgroup of O_1^s , where O_1 acts on each \mathbf{R}^2 via the antipodal map. Observe that the map $\Phi_{\mathbf{R}}$ is a "small cover" in the sense of [DJ]. The branched loci are $E_{m-1}{}^m\widetilde{\mathcal{P}}^2$ and $E_{m-1}{}^m\mathcal{P}_+^2$ respectively. As for ${}^m\mathcal{P}^2$ we have to add the branched locus ${}^m\mathcal{P}^1$. The generic points of ${}^m\mathcal{P}^1$ have a neighbourhood modelled on the quotient of \mathbf{C}^{m-2} by complex conjugation.

Analogously, the map $\Phi \colon \mathbf{G}_2(\mathbf{C}^m) \longrightarrow {}^m \mathcal{P}^3$ gives rise, for the space ${}^m \widetilde{\mathcal{P}}^3$, to a smooth *complex orbifold structure*. By that we mean a space locally modelled on the quotient of \mathbf{C}^s by a subgroup of U_1^s . We define the space $\mathcal{C}^{\infty}({}^m \mathcal{P}^3)$ of *smooth* maps from ${}^m \mathcal{P}^3$ to the reals as the subspace of $\mathcal{C}^{\infty}(\mathbf{G}_2(\mathbf{C}^m))$ which is invariant by the action of U_1^m .

(3.13) Riemannian and Poisson structures. Let $\mathcal{H}(m)$ be the space of Hermitian $(m \times m)$ -matrices, identified with \mathbf{u}_m^* via the pairing

$$\mathcal{H}(m) \times \mathbf{u_m} \longrightarrow \mathbf{R} \qquad (H, X) \mapsto \frac{i}{2} \operatorname{tr}(HX).$$

This identification turns the co-adjoint action of U_m into the conjugation action on $\mathcal{H}(m)$. Consider the map $\widetilde{\Psi}: \mathcal{M}_{m\times 2}(\mathbf{C}) \longrightarrow \mathcal{H}(m)$ given by $\widetilde{\Psi}(a,b) := (a,b)\cdot (a,b)^*$. One has $\widetilde{\Psi}\big(Q\cdot (a,b)\cdot P\big) = Q\cdot \widetilde{\Psi}((a,b)\cdot Q^*)$ for $P\in U_2$ and $Q\in U_m$ and thus $\mathcal{C}:=\widetilde{\Psi}\big(\mathbf{V}_2(\mathbf{C}^m)\big)$ is the U_m -orbit through $\mathrm{diag}(1,1,0,\ldots,0)$. This proves that $\widetilde{\Psi}$ descends to a diffeomorphism $\Psi:\mathbf{G}_2(\mathbf{C}^m)\stackrel{\simeq}{\longrightarrow} \mathcal{C}$.

The complex vector space $\mathcal{M}_{m\times 2}(\mathbb{C})$ is endowed with its classical Hermitian structure $\langle A,B\rangle:=\operatorname{tr}(AB^*)$, with associated symplectic form $\omega(\,,\,)=-\operatorname{Im}\langle\,,\,\rangle$. The map $\widetilde{\Psi}$ above and the map $\widetilde{\Phi}:\mathcal{M}_{m\times 2}(\mathbb{C})\longrightarrow \mathcal{H}_0(2)$ given by

$$\widetilde{\Phi}(a,b) := (a,b)^* \cdot (a,b) - \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

are moment maps for the Hamiltonian actions of U_m and U_2 respectively. One has $\mathbf{V}_2(\mathbf{C}^m) = \widetilde{\Phi}^{-1}(0)$ and thus $\mathbf{G}_2(\mathbf{C}^m)$ occurs as symplectic reduction of the Hermitian vector space $\mathcal{M}_{m\times 2}(\mathbf{C})$ and thereby inherits a U_m -invariant Kähler structure, using, for instance [Ki], §1.7. (Strictly speaking, one deals in [Ki] with compact Kähler manifolds; to fulfill this condition, one can first divide $\mathcal{M}_{m\times 2}(\mathbf{C}) - \{0\}$ by the diagonal action of \mathbf{C}^* to put oneself into a complex projective space.) The residual map $\Psi: \mathbf{G}_2(\mathbf{C}^m) \xrightarrow{\simeq} \mathcal{C} \subset \mathcal{H}(m)$ is a moment map for the action of U_m on $\mathbf{G}_2(\mathbf{C}^m)$.

Being thus a Kähler manifold, $G_2(\mathbb{C}^m)$ is a Riemannian Poisson manifold. This structure descends to the complex orbifold ${}^m\mathcal{P}^3$: the algebra $\mathcal{C}^{\infty}({}^m\mathcal{P}^3)$ admits a unique Lie bracket so that the projection $G_2(\mathbb{C}^m) \longrightarrow {}^m\mathcal{P}^3$ is a Poisson map.

(3.14) It is possible to endow with a Poisson structure the space ${}^m \mathcal{PP}_+^3$ of configurations of all m-gons in \mathbb{R}^3 , without fixing the perimeter to 2. It suffices in the above construction, to replace the U_2 -reduction $G_2(\mathbb{C}^m) = \widetilde{\Phi}^{-1}(0)/U_2$ by the SU_2 -reduction $\widetilde{G}_2(\mathbb{C}^m) := \widetilde{\Phi}^{-1}(0)/SU_2$. The latter is a non-compact space, the total space of the determinant bundle over $G_2(\mathbb{C}^m)$ with the zero

section collapsed. The trace function on $\mathcal{M}_{m\times 2}(\mathbf{C})$ descends to $\widetilde{\mathbf{G}}_2(\mathbf{C}^m)$ and to the Casimir function "perimeter" on ${}^m\mathcal{PP}_+^3$.

4. POLYGONS WITH GIVEN SIDES - KÄHLER STRUCTURES

We now use the map $\ell: {}^m\widetilde{\mathcal{P}}^k, {}^m\mathcal{P}^k_+, {}^m\mathcal{P}^k \to \mathbf{R}^m$ defined in (2.4). Recall that $\ell(\rho)$, for $\rho \in {}^m\widetilde{\mathcal{P}}^k$, is the length of the successive sides of a representative of r with total perimeter 2.

For
$$\alpha = (\alpha_1, \dots, \alpha_m) \in \mathbf{R}_{\geq 0}^m$$
 with $\sum_{i=1}^m \alpha_i = 2$, we define

$${}^{m}\widetilde{\mathcal{P}}^{k}(\alpha) := : \widetilde{\mathcal{P}}^{k}(\alpha) := \{ \rho \in {}^{m}\widetilde{\mathcal{P}}^{k} \mid \ell(\rho) = \alpha \} \subset {}^{m}\widetilde{\mathcal{P}}^{k} .$$

The space $\widetilde{\mathcal{P}}^k(\alpha)$ is invariant under the action of O_k . We define the moduli spaces

$$\mathcal{P}_{+}^{k}(\alpha) := SO_{k} \setminus \widetilde{\mathcal{P}}^{k}(\alpha) = \ell^{-1}(\alpha) \subset {}^{m}\mathcal{P}_{+}^{k}$$

and

$$\mathcal{P}^k(\alpha) := O_k \setminus \widetilde{\mathcal{P}}^k(\alpha) = \ell^{-1}(\alpha) \subset {}^m \mathcal{P}^k.$$

The space $\widetilde{\mathcal{P}}^1(\alpha)$ consists of a finite number of points and is generically empty. We call α generic if $\widetilde{\mathcal{P}}^1(\alpha) = \emptyset$.

THEOREM 4.1. The map $\mu := \ell \circ \widehat{\Phi} : \mathbf{G}_2(\mathbf{C}^m) \longrightarrow \mathbf{R}^m$ is a moment map for the action of U_1^m on $\mathbf{G}_2(\mathbf{C}^m)$.

Proof. As seen in (3.13), the moment map $\Psi: \mathbf{G}_2(\mathbf{C}^m) \longrightarrow \mathcal{H}(m)$ for the U_m -action on $\mathbf{G}_2(\mathbf{C}^m)$ is induced from $\widetilde{\Psi}: \mathcal{M}_{m \times 2}(\mathbf{C}) \longrightarrow \mathcal{H}(m)$ given by $\widetilde{\Psi}(a,b) := (a,b) \cdot (a,b)^*$. A moment map μ for the action of U_1^m is obtained by composing Ψ with the projection $\mathcal{H}(m) \longrightarrow \mathbf{R}^m$ associating to a matrix its diagonal entries. So, if $\Pi \in \mathbf{G}_2(\mathbf{C}^m)$ is generated by a and b with $(a,b) \in \mathbf{V}_2(\mathbf{C}^m)$, one has

$$\mu(\Pi) = (|a_1|^2 + |b_1|^2, \dots, |a_m|^2 + |b_m|^2) = \ell \circ \widehat{\Phi}(a, b).$$

A now classic theorem of Atiyah and Guillemin-Sternberg [Au, § III.4.2] asserts that the image of a moment map for a torus action is a convex polytope (the *moment polytope*). The restriction of the moment map to the fixed point set of an anti-symplectic involution has the same image [Du]. In our case, one gets these facts directly: