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calculate the quadrilateral, pentagon and hexagon spaces. Section 7 lists some open problems.

The study of the polygon spaces will be pursued in a forthcoming paper [HK] in which we shall compute the cohomology ring of these spaces.

The first author was incited by Sylvain Cappell to introduce symplectic geometry in his study of polygon spaces. He is also grateful to Lisa Jeffrey and Michèle Audin for useful conversations. The two authors started this work at the workshop in symplectic geometry organized in Cambridge by the Isaac Newton Institute (Fall 1994). The second author would like to thank Richard Montgomery for teaching him about dual pairs, and Michael Thaddeus for pointing out the link to moduli spaces of flat connections; also the University of Geneva for its hospitality while this paper was being written.

2. THE POLYGON SPACES

(2.1) Let V be a real vector space and m a positive integer. Let ${}^m\mathcal{F}(V)$ be the real vector space of all maps $\rho: \{1, 2, \dots, m\} \rightarrow V$ such that $\sum_{j=1}^m \rho(j) = 0$. An element $\rho \in {}^m\mathcal{F}(V)$ will be regarded as a closed polygonal path in V

$$0 \bullet \rightarrow \bullet \rho(1) \bullet \rightarrow \bullet \rho(1) + \rho(2) \bullet \rightarrow \bullet \dots \bullet \rightarrow \bullet \sum_{j=1}^m \rho(j) = 0$$

of m steps, or, alternately, as a configuration in V (up to translation) of a polygon of m sides. We shall call an element $\rho \in {}^m\mathcal{F}(V)$ an m -*polygon* (in V) and a *proper polygon* when $\rho(j) \neq 0 \ \forall j$. We use the notation ${}^m\mathcal{F}^k$ for the space ${}^m\mathcal{F}(\mathbf{R}^k)$.

The group \mathbf{R}_+ of positive homotheties of V acts freely and properly on ${}^m\mathcal{F}(V) - \{0\}$. The quotient ${}^m\tilde{\mathcal{P}}(V) := ({}^m\mathcal{F}(V) - \{0\})/\mathbf{R}_+$ then inherits a structure of smooth manifold diffeomorphic to a sphere. For instance, ${}^m\tilde{\mathcal{P}}^k := ({}^m\mathcal{F}^k - \{0\})/\mathbf{R}_+$ is diffeomorphic to the sphere $S^{k(m-1)-1}$.

(2.2) Suppose now that V is oriented and is a Euclidean space, namely V is endowed with a scalar product. The group $O(V)$ of isometries of V acts on ${}^k\mathcal{F}^m$ and ${}^m\tilde{\mathcal{P}}(V)$; we define the moduli spaces

$${}^m\mathcal{P}(V)_+ := SO(V) \backslash {}^m\tilde{\mathcal{P}}(V) \quad \text{and} \quad {}^m\mathcal{P}(V) := O(V) \backslash {}^m\tilde{\mathcal{P}}(V)$$

of m -polygons in V , up to similitude (where $SO(V)$ is the identity component of $O(V)$). Observe that any orientation preserving isometry $h: V \xrightarrow{\sim} \mathbf{R}^k$ produces identifications

$${}^m\mathcal{P}(V)_+ \simeq {}^m\mathcal{P}_+^k := SO_k \backslash {}^m\tilde{\mathcal{P}}^k \quad \text{and} \quad {}^m\mathcal{P}(V) \simeq {}^m\mathcal{P}^k := O_k \backslash {}^m\tilde{\mathcal{P}}^k.$$

We shall use the fact that these identifications do not depend on the choice of h and thus ${}^m\mathcal{P}(V)_+$ and ${}^m\mathcal{P}(V)$, for any Euclidean space V , are canonically identified with ${}^m\mathcal{P}_+^k$ and ${}^m\mathcal{P}^k$.

(2.3) The “degree of improperness” of polygons provides a stratification

$$\emptyset = E_1 {}^m\tilde{\mathcal{P}}(V) \subset E_2 {}^m\tilde{\mathcal{P}}(V) \subset \cdots \subset E_{m-1} {}^m\tilde{\mathcal{P}}(V) \subset E_m {}^m\tilde{\mathcal{P}}(V) = {}^m\tilde{\mathcal{P}}(V)$$

where

$$E_j {}^m\tilde{\mathcal{P}}(V) := \{\rho \in {}^m\tilde{\mathcal{P}}(V) \mid \#\{s \mid \rho(s) = 0\} \geq m - j\}.$$

The “open stratum” $E_j {}^m\tilde{\mathcal{P}}(V) - E_{j-1} {}^m\tilde{\mathcal{P}}(V)$ is a smooth submanifold of ${}^m\tilde{\mathcal{P}}(V)$ of dimension $(j-1)k-1$ if $k = \dim V$. The top open stratum ${}^m\tilde{\mathcal{P}}(V) - E_{m-1} {}^m\tilde{\mathcal{P}}(V)$, open and dense in ${}^m\tilde{\mathcal{P}}(V)$, is the space of proper polygons.

As this stratification is $O(V)$ -invariant, it projects onto stratifications $\{E_j {}^m\mathcal{P}_+^k\}$ and $\{E_j {}^m\mathcal{P}^k\}$ of the moduli spaces (using the canonical identifications of (2.2)). We shall see in §3 that the above stratifications describe the singular loci of smooth orbifold structures on the spaces ${}^m\tilde{\mathcal{P}}(V)$, ${}^m\mathcal{P}_+^k$ and ${}^m\mathcal{P}^k$.

(2.4) The map $\rho \mapsto |\rho| := \sum_{j=1}^m |\rho(j)|$ which associates to a polygon ρ its total perimeter is a norm on ${}^m\mathcal{F}(V)$. We denote by $S({}^m\mathcal{F}(V))$ the sphere of radius 2 for this norm. Each class in ${}^m\tilde{\mathcal{P}}(V)$ has a unique representative in $S({}^m\mathcal{F}(V))$ which gives a topological embedding $\iota: {}^m\tilde{\mathcal{P}}(V) \rightarrow S({}^m\mathcal{F}(V))$ whose image is $S({}^m\mathcal{F}(V))$. The image by ι of $E_{m-1} {}^m\tilde{\mathcal{P}}(V)$ is the subset of $S({}^m\mathcal{F}(V))$ where $S({}^m\mathcal{F}(V))$ fails to be a smooth submanifold of ${}^m\mathcal{F}(V)$. However, the restriction of ι to each $E_j {}^m\tilde{\mathcal{P}}(V) - E_{j-1} {}^m\tilde{\mathcal{P}}(V)$ is a smooth embedding.

The map $\tilde{\ell}: {}^m\mathcal{F}(V) \rightarrow \mathbf{R}^m$ defined by $\tilde{\ell}(\rho) := (|\rho(1)|, \dots, |\rho(m)|)$ associates to a polygon its side-lengths. We define $\ell: {}^m\tilde{\mathcal{P}}(V) \rightarrow \mathbf{R}^m$ by $\ell := \tilde{\ell} \circ \iota$. We shall also use the notation $\ell_i(\rho)$ for $|\rho(i)|$. These maps are invariant under the $O(V)$ -action and thus define maps (always called ℓ)

$$\ell: {}^m\mathcal{P}_+^k \rightarrow \mathbf{R}^m \quad \text{and} \quad \ell: {}^m\mathcal{P}^k \rightarrow \mathbf{R}^m$$

which are smooth on each open stratum.

(2.5) Let $\tau: V \rightarrow V$ be the orthogonal reflection through some hyperplane Π in V . One has the involution $\rho \mapsto \check{\rho} := \tau \circ \rho$ on ${}^m\mathcal{F}(V)$ and ${}^m\tilde{\mathcal{P}}(V)$ whose fixed-point space is naturally ${}^m\mathcal{F}(\Pi)$ and ${}^m\tilde{\mathcal{P}}(\Pi)$. If $h \in SO(V)$, one has

$$\tau \circ h = \underbrace{(\tau \circ h \circ \tau \circ h^{-1})}_{\in SO(V)} \circ h \circ \tau.$$

Hence the involution descends to an involution (still denoted $\rho \mapsto \check{\rho}$) on ${}^m\mathcal{P}_+^k$. If τ' is an orthogonal reflection with respect to another hyperplane Π' , then the formula $\tau \circ \rho' = (\tau' \circ \tau) \circ \tau \circ \rho$ shows that the induced involution on ${}^m\mathcal{P}_+^k$ does not depend on the choice of τ . The fixed point space of $\check{\cdot}$ is ${}^m\mathcal{P}^{k-1}$. Observe that $\check{\rho} = \rho$ in ${}^m\mathcal{P}^k$.

EXAMPLES

(2.6) *Polygons in the line.* The space ${}^m\tilde{\mathcal{P}}^1 = {}^m\tilde{\mathcal{P}}_+^1$ is diffeomorphic to the sphere S^{m-2} . Under this identification, the O_1 -action becomes the antipodal map and thus ${}^m\mathcal{P}^1$ is a smooth manifold diffeomorphic to $\mathbf{R}P^{m-2}$. For example, ${}^3\tilde{\mathcal{P}}^1 \simeq S^1$ and ${}^3\mathcal{P}^1 \simeq \mathbf{R}P^1$. The stratum $E_2{}^3\tilde{\mathcal{P}}^1$ consists of 3 pairs of antipodal points and thus $E_2{}^3\mathcal{P}^1$ is a set of 3 points, the three triangles with one side of length 0. This corresponds to the fact that $S({}^3\mathcal{F}^1)$ is a regular hexagon and $O_1 \backslash S({}^3\mathcal{F}^1)$ is a triangle. Actually, the map $\ell : {}^3\mathcal{P}^1 \longrightarrow \mathbf{R}^3$ produces homeomorphisms

$${}^3\mathcal{P}^1 \xrightarrow[\simeq]{\ell} \{(x, y, z) \in \mathbf{R}_{\geq 0}^3 \mid x + y + z = 2 \text{ and } \pm x \pm y \pm z = 0\}.$$

(2.7) *Polygons in the plane.* Identifying \mathbf{R}^2 with \mathbf{C} , the space ${}^m\mathcal{F}^2$ is a complex vector space isomorphic to \mathbf{C}^{m-1} and the (free) SO_2 -action corresponds to the diagonal U_1 -action. As in (2.6) one establishes the diffeomorphisms

$$\begin{array}{ccc} {}^m\tilde{\mathcal{P}}^2 & \xrightarrow{\simeq} & S^{2m-3} \\ \downarrow & & \downarrow \\ {}^m\mathcal{P}_+^2 & \xrightarrow{\simeq} & \mathbf{C}P^{m-2} \end{array}$$

The above diffeomorphisms conjugate the involutions $\check{\cdot}$ with the complex conjugations of \mathbf{C}^{m-1} and $\mathbf{C}P^{m-2}$. Also, the involution $\check{\cdot}$ on ${}^m\mathcal{P}_+^2$ coincides with the residual O_1 action and therefore ${}^m\mathcal{P}^2$ is the quotient of $\mathbf{C}P^{m-2}$ by its complex conjugation.

For example, ${}^3\tilde{\mathcal{P}}^2$, the space of planar triangles, is diffeomorphic to the sphere S^3 . The singular stratum $E_2({}^3\tilde{\mathcal{P}}^1)$ is a link of three circles which are SO_2 -orbits (therefore, any two of them constitute a Hopf link). The quotient ${}^3\mathcal{P}_+^2$ is identified with $\mathbf{C}P^1$ and $E_2({}^3\tilde{\mathcal{P}}_+^1)$ is a set of three points in $\mathbf{C}P^1$.

Finally, ${}^3\mathcal{P}^2 \simeq \mathbf{CP}^1 / \{z \sim \bar{z}\}$ is homeomorphic, via the length-side map ℓ , to the solid triangle

$${}^3\mathcal{P}^2 = {}^3\mathcal{P}^3 \xrightarrow[\simeq]{\ell} \{(x_1, x_2, x_3) \in \mathbf{R}^3 \mid x_1 + x_2 + x_3 = 2 \text{ and } 0 \leq x_i \leq 1\}$$

with boundary ${}^3\mathcal{P}^1$.

3. QUATERNIONS, GRASSMANNIANS AND STRUCTURES ON THE FULL POLYGON SPACES

(3.1) Let $\mathbf{H} = \mathbf{C} \oplus \mathbf{C}j$ be the skew-field of quaternions; the space $I\mathbf{H}$ of pure imaginary quaternions is equipped with the orthonormal basis i, j and $k = ij$, giving rise to an isometry with \mathbf{R}^3 which turns the pure imaginary part of the quaternionic multiplication pq into the usual cross product $p \times q$. The space ${}^m\mathcal{F}^3$ is thus identified with ${}^m\mathcal{F}(I\mathbf{H})$ which gives rise to the canonical identifications on the various moduli spaces (see (2.2)).

Recall that the correspondence

$$\eta : u + vj \mapsto \begin{pmatrix} u & v \\ -\bar{v} & \bar{u} \end{pmatrix}$$

gives an injective \mathbf{R} -algebra homomorphism $\eta : \mathbf{H} \longrightarrow \mathcal{M}_{(2 \times 2)}(\mathbf{C})$. This enables a matrix $P \in U_2$ to act on the right or on the left on \mathbf{H} . It also identifies the group S^3 of unit quaternions with SU_2 .

(3.2) The Hopf map $\phi : \mathbf{H} \longrightarrow I\mathbf{H}$ defined by

$$\phi(q) := \bar{q} i q$$

sends the 3-sphere of radius \sqrt{r} in \mathbf{H} onto the 2-sphere of radius r in $I\mathbf{H}$. (The formulae given in the original paper by Hopf [Ho, §5] actually correspond to the map $q \mapsto \bar{q} k q$.) The equality $\phi(q) = \phi(q')$ occurs if and only if $q' = e^{i\theta} q$. The map ϕ satisfies the equivariance relation $\phi(q \cdot P) = P^{-1} \cdot \phi(q) \cdot P$. Writing $q = u + vj$ with $u, v \in \mathbf{C}$, one has

$$\phi(u + vj) = (\bar{u} - j\bar{v}) i (u + vj) = i(\bar{u} + j\bar{v})(u + vj) = i[|u|^2 - |v|^2 + 2\bar{u}vj].$$

(3.3) Observe that if $q = s + tj$ with $s, t \in \mathbf{R}$, then $\phi(q) = i q^2$. This plane $\mathbf{R} \oplus \mathbf{R}j$ of its images is the fixed point set of the involution $a + bj \mapsto \bar{a} + \bar{b}j$ that will be used later. Its image under ϕ is $\mathbf{R}i \oplus \mathbf{R}k$.

(3.4) REMARK. $I\mathbf{H}$, with the Lie bracket $[p, q] = pq - qp = 2 \operatorname{Im}(pq)$, is the Lie algebra for the group $U_1(\mathbf{H}) \simeq SU_2 \simeq S^3$. The pairing