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$$\begin{aligned}\Phi(A^t) &= \Phi \left( \left( \begin{array}{cc} 1 & b(e^{-2t} - 1)/2 \\ 0 & 1 \end{array} \right) \left( \begin{array}{cc} e^{-t} & 0 \\ 0 & e^t \end{array} \right) \right) \\ &= \Phi \left( \left( \begin{array}{cc} 1 & b(e^{-2t} - 1)/2 \\ 0 & 1 \end{array} \right) \right) \left( \begin{array}{cc} e^{-t} & 0 \\ 0 & e^t \end{array} \right)\end{aligned}$$

since  $H$  acts linearly on the  $x$ -axis. Hence, since the family of maps

$$F_t = \Phi \left( \left( \begin{array}{cc} 1 & b(e^{-2t} - 1)/2 \\ 0 & 1 \end{array} \right) \right) \quad t \geq 0$$

is equicontinuous in some neighbourhood of the identity, we conclude that  $\Sigma_A$  is the  $x$ -axis, as required.

By the above argument, we may assume that locally the  $SO(2)$ -action is the canonical one and the  $SL(2, \mathbf{R})$ -action preserves the radial lines. The proof is then completed as in the proof of Theorem 4.2.  $\square$

## 7. EXAMPLES OF $C^0$ -ACTIONS OF $SL(2, \mathbf{R})$ ON $\mathbf{R}^m$

When  $m$  is greater than  $n$  there is a plethora of examples of continuous actions of  $SL(n, \mathbf{R})$  on  $(\mathbf{R}^m, 0)$ . In this section we give some examples in the case  $n = 2$ .

7.1. THE SYMMETRIC PRODUCT. Choose one of the continuous  $SL(2, \mathbf{R})$ -actions on  $(\mathbf{R}^2, 0)$  from the previous section. Now consider the associated  $SL(2, \mathbf{R})$ -action on the symmetric product

$$\Pi_{i=1}^m \mathbf{R}^2 / \Sigma_m \cong \mathbf{C}^m,$$

where  $\Sigma_m$  is the symmetric group on  $m$  letters. Recall that the last identification associates to an  $m$ -tuple of points  $(x_1, \dots, x_m)$  in  $\mathbf{R}^2 \cong \mathbf{C}$  the coefficients of the monic polynomial of degree  $m$  in one complex variable whose roots are the  $x_i$ . As the original action fixed the origin in  $\mathbf{R}^2$ , so the corresponding action fixes the origin in  $\mathbf{R}^{2m}$ .

7.2. THE ADJOINT ACTION AT INFINITY. Consider the adjoint action of  $SL(2, \mathbf{R})$  on  $\mathbf{R}^3$ , as discussed in Section 5. Removing the origin and compactifying the other end, we obtain a  $C^0$ -action of  $SL(2, \mathbf{R})$  on  $\mathbf{R}^3$ , which we will call the *adjoint action at infinity*. This action is certainly not topologically linearizable, since all the orbits now accumulate to the fixed point. In fact, this action is not topologically conjugate to any  $C^1$ -action. To see this, consider the hyperbolic element  $h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ . Using the exponential

$\exp(th)$ , one obtains a one-parameter subgroup in  $SL(2, \mathbf{R})$  which, by the adjoint action, defines a flow  $\mathfrak{F}$  on  $\mathfrak{sl}(2, \mathbf{R})$ . Choose the following basis for  $\mathfrak{sl}(2, \mathbf{R}) \cong \mathbf{R}^3$ :

$$\left\{ \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \right\}.$$

Then a simple computation shows that the flow  $\mathfrak{F}$  is generated by the vector field  $X = z \frac{\partial}{\partial y} + y \frac{\partial}{\partial z}$  (where  $(x, y, z)$  are the coordinates with respect to the above basis). Restricted to each plane  $x = \text{constant}$ , the vector field  $X$  has a standard hyperbolic singularity, with index  $-1$ , and on the invariant lines  $z = -y$  and  $z = y$ , the flow is contracting and expanding respectively. It follows that if the  $SL(2, \mathbf{R})$ -action at infinity was  $C^1$ , then the differential at infinity of the action of  $X$  would be trivial. In this case, the differential at infinity of the entire  $SL(2, \mathbf{R})$ -action would be trivial, contradicting Thurston's stability theorem.

7.3. THE ACTION ON THE CLOSED SUBGROUPS OF  $\mathbf{R}^2$ . Recall that from [35] the space  $Gr$  of closed subgroups of  $\mathbf{R}^2$ , with the Hausdorff topology, is homeomorphic to  $S^4$ . Obviously  $SL(2, \mathbf{R})$  acts continuously on  $Gr$ , and the two trivial subgroups,  $\{0\}$  and  $\mathbf{R}^2$ , are fixed by this action. Inside  $Gr$  there is an invariant  $S^3$  comprised of the set  $K$  of subgroups isomorphic to  $\mathbf{R}$ , together with the set of subgroups isomorphic to  $\mathbf{Z}^2$  which have generators which span a parallelogram of area 1. The set  $K$ , which is a trefoil knot in  $S^3$ , is a 1-dimensional orbit, and its complement  $S^3 - K$  is a single 3-dimensional orbit.

Removing one of the fixed subgroups,  $\{0\}$  or  $\mathbf{R}^2$ , one obtains an interesting  $SL(2, \mathbf{R})$ -action on  $\mathbf{R}^4$  with one fixed point. Notice that this action is not conjugate to a  $C^1$ -action. Indeed, if the action was  $C^1$ , then the differential at the origin would define a linear representation of  $SL(2, \mathbf{R})$  in  $\mathbf{R}^4$ . So this representation would be a direct sum of irreducible representations. Since  $-\text{Id}$  acts trivially on  $Gr$ , it follows that it is either the sum of the canonical 3-dimensional representation with the trivial 1-dimensional representation, or it is the trivial 4-dimensional representation. But the second case is not possible, by Thurston's stability theorem. In the first case, one could linearize the  $SO(2)$ -action, using the Bochner-Cartan theorem, and thus locally one would find a 2-dimensional subspace through the origin which was fixed pointwise by  $SO(2)$ . But there are no closed subgroups of  $\mathbf{R}^2$  which are  $SO(2)$ -invariant, apart from  $\{0\}$  and  $\mathbf{R}^2$ . So this case is also impossible.

7.4. CONING ACTIONS ON SPHERES. If one has a non-trivial  $SL(2, \mathbf{R})$ -action on  $S^m$ , then taking the cone in the obvious sense, one obtains an  $SL(2, \mathbf{R})$ -action on  $(\mathbf{R}^{m+1}, 0)$ . We claim that such actions cannot be conjugate to  $C^1$  actions. Indeed, actions defined by coning have invariant spheres around 0. If a  $C^1$  diffeomorphism has a family of invariant topological spheres around the origin, it cannot have any stable manifold so that all the eigenvalues of its differential at the origin have modulus one. No non-trivial linear representation of  $SL(2, \mathbf{R})$  has the property that all eigenvalues of all elements have modulus one. So, if the action under consideration was  $C^1$  the differential at the origin would be trivial: this is a contradiction with Thurston's stability theorem.

There are many interesting actions of  $SL(2, \mathbf{R})$  on spheres. Compactifying the actions of Section 6 gives examples on  $S^2$ . An action on  $S^3$  was given in Example 7.3. Notice also that if one has actions of  $SL(2, \mathbf{R})$  on  $S^p$  and  $S^q$ , then there is an associated action of  $SL(2, \mathbf{R})$  on their join  $S^p * S^q = S^{p+q+1}$ .

Finally we remark that many interesting actions of  $SL(n, \mathbf{R})$  on spheres, for  $n \geq 3$ , can be found in the papers of Fuichi Uchida (see for example [46, 47, 48]).

## 8. A $C^\infty$ -ACTION OF $SL(2, \mathbf{R})$ WHICH IS NOT LINEARIZABLE

Here we give a variation of the Guillemin-Sternberg example a  $C^\infty$ -action of the Lie algebra  $\mathfrak{sl}(2, \mathbf{R})$  on  $\mathbf{R}^3$  which is not linearizable. The action we give below integrates to a  $C^\infty$  non-linearizable  $SL(2, \mathbf{R})$ -action. It is obtained by deforming the adjoint action of  $SL(2, \mathbf{R})$  on its Lie algebra. The constructed action is clearly non-linearizable since it has an orbit of dimension 3.

By differentiation, the adjoint action of  $SL(2, \mathbf{R})$  defines a Lie algebra  $\mathfrak{g}$  (isomorphic to  $\mathfrak{sl}(2, \mathbf{R})$ ) of vector fields on  $\mathbf{R}^3$ . This algebra can be explicitly computed as follows: choose an element  $h \in \mathfrak{sl}(2, \mathbf{R})$ , take its exponential  $\exp h$ , and compute the derivative of the adjoint map  $Ad(\exp(th))$  at  $t = 0$ . A convenient basis for  $\mathfrak{g}$  is:

$$X = z \frac{\partial}{\partial y} + y \frac{\partial}{\partial z}, \quad Y = z \frac{\partial}{\partial x} + x \frac{\partial}{\partial z}, \quad R = x \frac{\partial}{\partial y} - y \frac{\partial}{\partial x}.$$

Here  $R$  is the derivative of  $Ad(\exp(th))$  where  $h = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ . The commutator relations are:

$$[X, Y] = -R, \quad [R, X] = Y, \quad [R, Y] = -X.$$