

Zeitschrift: L'Enseignement Mathématique
Herausgeber: Commission Internationale de l'Enseignement Mathématique
Band: 43 (1997)
Heft: 1-2: L'ENSEIGNEMENT MATHÉMATIQUE

Artikel: THE LOCAL LINEARIZATION PROBLEM FOR SMOOTH $SL(n)$ - ACTIONS
Autor: CAIRNS, Grant / Ghys, Étienne
Kapitel: 7. Examples of $SL(2, \mathbb{R})$ -actions of $SL(2, \mathbb{R})$ on \mathbb{R}^m
DOI: <https://doi.org/10.5169/seals-63275>

Nutzungsbedingungen

Die ETH-Bibliothek ist die Anbieterin der digitalisierten Zeitschriften auf E-Periodica. Sie besitzt keine Urheberrechte an den Zeitschriften und ist nicht verantwortlich für deren Inhalte. Die Rechte liegen in der Regel bei den Herausgebern beziehungsweise den externen Rechteinhabern. Das Veröffentlichen von Bildern in Print- und Online-Publikationen sowie auf Social Media-Kanälen oder Webseiten ist nur mit vorheriger Genehmigung der Rechteinhaber erlaubt. [Mehr erfahren](#)

Conditions d'utilisation

L'ETH Library est le fournisseur des revues numérisées. Elle ne détient aucun droit d'auteur sur les revues et n'est pas responsable de leur contenu. En règle générale, les droits sont détenus par les éditeurs ou les détenteurs de droits externes. La reproduction d'images dans des publications imprimées ou en ligne ainsi que sur des canaux de médias sociaux ou des sites web n'est autorisée qu'avec l'accord préalable des détenteurs des droits. [En savoir plus](#)

Terms of use

The ETH Library is the provider of the digitised journals. It does not own any copyrights to the journals and is not responsible for their content. The rights usually lie with the publishers or the external rights holders. Publishing images in print and online publications, as well as on social media channels or websites, is only permitted with the prior consent of the rights holders. [Find out more](#)

Download PDF: 19.08.2025

ETH-Bibliothek Zürich, E-Periodica, <https://www.e-periodica.ch>

$$\begin{aligned}\Phi(A^t) &= \Phi \left(\begin{pmatrix} 1 & b(e^{-2t} - 1)/2 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} e^{-t} & 0 \\ 0 & e^t \end{pmatrix} \right) \\ &= \Phi \left(\begin{pmatrix} 1 & b(e^{-2t} - 1)/2 \\ 0 & 1 \end{pmatrix} \right) \begin{pmatrix} e^{-t} & 0 \\ 0 & e^t \end{pmatrix}\end{aligned}$$

since H acts linearly on the x -axis. Hence, since the family of maps

$$F_t = \Phi \left(\begin{pmatrix} 1 & b(e^{-2t} - 1)/2 \\ 0 & 1 \end{pmatrix} \right) \quad t \geq 0$$

is equicontinuous in some neighbourhood of the identity, we conclude that Σ_A is the x -axis, as required.

By the above argument, we may assume that locally the $SO(2)$ -action is the canonical one and the $SL(2, \mathbf{R})$ -action preserves the radial lines. The proof is then completed as in the proof of Theorem 4.2. \square

7. EXAMPLES OF C^0 -ACTIONS OF $SL(2, \mathbf{R})$ ON \mathbf{R}^m

When m is greater than n there is a plethora of examples of continuous actions of $SL(n, \mathbf{R})$ on $(\mathbf{R}^m, 0)$. In this section we give some examples in the case $n = 2$.

7.1. THE SYMMETRIC PRODUCT. Choose one of the continuous $SL(2, \mathbf{R})$ -actions on $(\mathbf{R}^2, 0)$ from the previous section. Now consider the associated $SL(2, \mathbf{R})$ -action on the symmetric product

$$\Pi_{i=1}^m \mathbf{R}^2 / \Sigma_m \cong \mathbf{C}^m,$$

where Σ_m is the symmetric group on m letters. Recall that the last identification associates to an m -tuple of points (x_1, \dots, x_m) in $\mathbf{R}^2 \cong \mathbf{C}$ the coefficients of the monic polynomial of degree m in one complex variable whose roots are the x_i . As the original action fixed the origin in \mathbf{R}^2 , so the corresponding action fixes the origin in \mathbf{R}^{2m} .

7.2. THE ADJOINT ACTION AT INFINITY. Consider the adjoint action of $SL(2, \mathbf{R})$ on \mathbf{R}^3 , as discussed in Section 5. Removing the origin and compactifying the other end, we obtain a C^0 -action of $SL(2, \mathbf{R})$ on \mathbf{R}^3 , which we will call the *adjoint action at infinity*. This action is certainly not topologically linearizable, since all the orbits now accumulate to the fixed point. In fact, this action is not topologically conjugate to any C^1 -action. To see this, consider the hyperbolic element $h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. Using the exponential

$\exp(th)$, one obtains a one-parameter subgroup in $SL(2, \mathbf{R})$ which, by the adjoint action, defines a flow \mathfrak{F} on $\mathfrak{sl}(2, \mathbf{R})$. Choose the following basis for $\mathfrak{sl}(2, \mathbf{R}) \cong \mathbf{R}^3$:

$$\left\{ \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \right\}.$$

Then a simple computation shows that the flow \mathfrak{F} is generated by the vector field $X = z \frac{\partial}{\partial y} + y \frac{\partial}{\partial z}$ (where (x, y, z) are the coordinates with respect to the above basis). Restricted to each plane $x = \text{constant}$, the vector field X has a standard hyperbolic singularity, with index -1 , and on the invariant lines $z = -y$ and $z = y$, the flow is contracting and expanding respectively. It follows that if the $SL(2, \mathbf{R})$ -action at infinity was C^1 , then the differential at infinity of the action of X would be trivial. In this case, the differential at infinity of the entire $SL(2, \mathbf{R})$ -action would be trivial, contradicting Thurston's stability theorem.

7.3. THE ACTION ON THE CLOSED SUBGROUPS OF \mathbf{R}^2 . Recall that from [35] the space Gr of closed subgroups of \mathbf{R}^2 , with the Hausdorff topology, is homeomorphic to S^4 . Obviously $SL(2, \mathbf{R})$ acts continuously on Gr , and the two trivial subgroups, $\{0\}$ and \mathbf{R}^2 , are fixed by this action. Inside Gr there is an invariant S^3 comprised of the set K of subgroups isomorphic to \mathbf{R} , together with the set of subgroups isomorphic to \mathbf{Z}^2 which have generators which span a parallelogram of area 1. The set K , which is a trefoil knot in S^3 , is a 1-dimensional orbit, and its complement $S^3 - K$ is a single 3-dimensional orbit.

Removing one of the fixed subgroups, $\{0\}$ or \mathbf{R}^2 , one obtains an interesting $SL(2, \mathbf{R})$ -action on \mathbf{R}^4 with one fixed point. Notice that this action is not conjugate to a C^1 -action. Indeed, if the action was C^1 , then the differential at the origin would define a linear representation of $SL(2, \mathbf{R})$ in \mathbf{R}^4 . So this representation would be a direct sum of irreducible representations. Since $-\text{Id}$ acts trivially on Gr , it follows that it is either the sum of the canonical 3-dimensional representation with the trivial 1-dimensional representation, or it is the trivial 4-dimensional representation. But the second case is not possible, by Thurston's stability theorem. In the first case, one could linearize the $SO(2)$ -action, using the Bochner-Cartan theorem, and thus locally one would find a 2-dimensional subspace through the origin which was fixed pointwise by $SO(2)$. But there are no closed subgroups of \mathbf{R}^2 which are $SO(2)$ -invariant, apart from $\{0\}$ and \mathbf{R}^2 . So this case is also impossible.

7.4. CONING ACTIONS ON SPHERES. If one has a non-trivial $SL(2, \mathbf{R})$ -action on S^m , then taking the cone in the obvious sense, one obtains an $SL(2, \mathbf{R})$ -action on $(\mathbf{R}^{m+1}, 0)$. We claim that such actions cannot be conjugate to C^1 actions. Indeed, actions defined by coning have invariant spheres around 0. If a C^1 diffeomorphism has a family of invariant topological spheres around the origin, it cannot have any stable manifold so that all the eigenvalues of its differential at the origin have modulus one. No non-trivial linear representation of $SL(2, \mathbf{R})$ has the property that all eigenvalues of all elements have modulus one. So, if the action under consideration was C^1 the differential at the origin would be trivial: this is a contradiction with Thurston's stability theorem.

There are many interesting actions of $SL(2, \mathbf{R})$ on spheres. Compactifying the actions of Section 6 gives examples on S^2 . An action on S^3 was given in Example 7.3. Notice also that if one has actions of $SL(2, \mathbf{R})$ on S^p and S^q , then there is an associated action of $SL(2, \mathbf{R})$ on their join $S^p * S^q = S^{p+q+1}$.

Finally we remark that many interesting actions of $SL(n, \mathbf{R})$ on spheres, for $n \geq 3$, can be found in the papers of Fuichi Uchida (see for example [46, 47, 48]).

8. A C^∞ -ACTION OF $SL(2, \mathbf{R})$ WHICH IS NOT LINEARIZABLE

Here we give a variation of the Guillemin-Sternberg example a C^∞ -action of the Lie algebra $\mathfrak{sl}(2, \mathbf{R})$ on \mathbf{R}^3 which is not linearizable. The action we give below integrates to a C^∞ non-linearizable $SL(2, \mathbf{R})$ -action. It is obtained by deforming the adjoint action of $SL(2, \mathbf{R})$ on its Lie algebra. The constructed action is clearly non-linearizable since it has an orbit of dimension 3.

By differentiation, the adjoint action of $SL(2, \mathbf{R})$ defines a Lie algebra \mathfrak{g} (isomorphic to $\mathfrak{sl}(2, \mathbf{R})$) of vector fields on \mathbf{R}^3 . This algebra can be explicitly computed as follows: choose an element $h \in \mathfrak{sl}(2, \mathbf{R})$, take its exponential $\exp h$, and compute the derivative of the adjoint map $Ad(\exp(th))$ at $t = 0$. A convenient basis for \mathfrak{g} is:

$$X = z \frac{\partial}{\partial y} + y \frac{\partial}{\partial z}, \quad Y = z \frac{\partial}{\partial x} + x \frac{\partial}{\partial z}, \quad R = x \frac{\partial}{\partial y} - y \frac{\partial}{\partial x}.$$

Here R is the derivative of $Ad(\exp(th))$ where $h = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. The commutator relations are:

$$[X, Y] = -R, \quad [R, X] = Y, \quad [R, Y] = -X.$$