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**ACTIONS** 

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$$\Phi(A^t) = \Phi\left(\begin{pmatrix} 1 & b(e^{-2t} - 1)/2 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} e^{-t} & 0 \\ 0 & e^t \end{pmatrix}\right)$$
$$= \Phi\left(\begin{pmatrix} 1 & b(e^{-2t} - 1)/2 \\ 0 & 1 \end{pmatrix} \right) \begin{pmatrix} e^{-t} & 0 \\ 0 & e^t \end{pmatrix}$$

since H acts linearly on the x-axis. Hence, since the family of maps

$$F_t = \Phi\left(\begin{pmatrix} 1 & b(e^{-2t} - 1)/2 \\ 0 & 1 \end{pmatrix}\right) \qquad t \ge 0$$

is equicontinuous in some neighbourhood of the identity, we conclude that  $\Sigma_A$  is the *x*-axis, as required.

By the above argument, we may assume that locally the SO(2)-action is the canonical one and the  $SL(2, \mathbf{R})$ -action preserves the radial lines. The proof is then completed as in the proof of Theorem 4.2.

# 7. Examples of $C^0$ -actions of $SL(2, \mathbf{R})$ on $\mathbf{R}^m$

When m is greater than n there is a plethora of examples of continuous actions of  $SL(n, \mathbf{R})$  on  $(\mathbf{R}^m, 0)$ . In this section we give some examples in the case n = 2.

7.1. THE SYMMETRIC PRODUCT. Choose one of the continuous  $SL(2, \mathbf{R})$ -actions on  $(\mathbf{R}^2, 0)$  from the previous section. Now consider the associated  $SL(2, \mathbf{R})$ -action on the symmetric product

$$\Pi_{i=1}^m \mathbf{R}^2 / \Sigma_m \cong \mathbf{C}^m$$
,

where  $\Sigma_m$  is the symmetric group on m letters. Recall that the last identification associates to an m-tuple of points  $(x_1, \ldots, x_m)$  in  $\mathbb{R}^2 \cong \mathbb{C}$  the coefficients of the monic polynomial of degree m in one complex variable whose roots are the  $x_i$ . As the original action fixed the origin in  $\mathbb{R}^2$ , so the corresponding action fixes the origin in  $\mathbb{R}^{2m}$ .

7.2. THE ADJOINT ACTION AT INFINITY. Consider the adjoint action of  $SL(2,\mathbf{R})$  on  $\mathbf{R}^3$ , as discussed in Section 5. Removing the origin and compactifying the other end, we obtain a  $C^0$ -action of  $SL(2,\mathbf{R})$  on  $\mathbf{R}^3$ , which we will call the *adjoint action at infinity*. This action is certainly not topologically linearizable, since all the orbits now accumulate to the fixed point. In fact, this action is not topologically conjugate to any  $C^1$ -action. To see this, consider the hyperbolic element  $h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ . Using the exponential

 $\exp(th)$ , one obtains a one-parameter subgroup in  $SL(2, \mathbf{R})$  which, by the adjoint action, defines a flow  $\mathfrak{F}$  on  $\mathfrak{sl}(2, \mathbf{R})$ . Choose the following basis for  $\mathfrak{sl}(2, \mathbf{R}) \cong \mathbf{R}^3$ :

$$\left\{ \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \right\}.$$

Then a simple computation shows that the flow  $\mathfrak{F}$  is generated by the vector field  $X = z \frac{\partial}{\partial y} + y \frac{\partial}{\partial z}$  (where (x, y, z) are the coordinates with respect to the above basis). Restricted to each plane x = constant, the vector field X has a standard hyperbolic singularity, with index -1, and on the invariant lines z = -y and z = y, the flow is contracting and expanding respectively. It follows that if the  $SL(2, \mathbf{R})$ -action at infinity was  $C^1$ , then the differential at infinity of the action of X would be trivial. In this case, the differential at infinity of the entire  $SL(2, \mathbf{R})$ -action would be trivial, contradicting Thurston's stability theorem.

7.3. THE ACTION ON THE CLOSED SUBGROUPS OF  $\mathbb{R}^2$ . Recall that from [35] the space Gr of closed subgroups of  $\mathbb{R}^2$ , with the Hausdorff topology, is homeomorphic to  $S^4$ . Obviously  $SL(2, \mathbb{R})$  acts continuously on Gr, and the two trivial subgroups,  $\{0\}$  and  $\mathbb{R}^2$ , are fixed by this action. Inside Gr there is an invariant  $S^3$  comprised of the set K of subgroups isomorphic to  $\mathbb{R}$ , together with the set of subgroups isomorphic to  $\mathbb{Z}^2$  which have generators which span a parallelogram of area 1. The set K, which is a trefoil knot in  $S^3$ , is a 1-dimensional orbit, and its complement  $S^3 - K$  is a single 3-dimensional orbit.

Removing one of the fixed subgroups,  $\{0\}$  or  $\mathbb{R}^2$ , one obtains an interesting  $SL(2,\mathbb{R})$ -action on  $\mathbb{R}^4$  with one fixed point. Notice that this action is not conjugate to a  $C^1$ -action. Indeed, if the action was  $C^1$ , then the differential at the origin would define a linear representation of  $SL(2,\mathbb{R})$  in  $\mathbb{R}^4$ . So this representation would be a direct sum of irreducible representations. Since —Id acts trivially on Gr, it follows that it is either the sum of the canonical 3-dimensional representation with the trivial 1-dimensional representation, or it is the trivial 4-dimensional representation. But the second case is not possible, by Thurston's stability theorem. In the first case, one could linearize the SO(2)-action, using the Bochner-Cartan theorem, and thus locally one would find a 2-dimensional subspace through the origin which was fixed pointwise by SO(2). But there are no closed subgroups of  $\mathbb{R}^2$  which are SO(2)-invariant, apart from  $\{0\}$  and  $\mathbb{R}^2$ . So this case is also impossible.

7.4. Coning actions on SPHERES. If one has a non-trivial  $SL(2, \mathbf{R})$ -action on  $S^m$ , then taking the cone in the obvious sense, one obtains an  $SL(2, \mathbf{R})$ -action on  $(\mathbf{R}^{m+1}, 0)$ . We claim that such actions cannot be conjugate to  $C^1$  actions. Indeed, actions defined by coning have invariant spheres around 0. If a  $C^1$  diffeomorphism has a family of invariant topological spheres around the origin, it cannot have any stable manifold so that all the eigenvalues of its differential at the origin have modulus one. No non-trivial linear representation of  $SL(2, \mathbf{R})$  has the property that all eigenvalues of all elements have modulus one. So, if the action under consideration was  $C^1$  the differential at the origin would be trivial: this is a contradiction with Thurston's stability theorem.

There are many interesting actions of  $SL(2, \mathbf{R})$  on spheres. Compactifying the actions of Section 6 gives examples on  $S^2$ . An action on  $S^3$  was given in Example 7.3. Notice also that if one has actions of  $SL(2, \mathbf{R})$  on  $S^p$  and  $S^q$ , then there is an associated action of  $SL(2, \mathbf{R})$  on their join  $S^p * S^q = S^{p+q+1}$ .

Finally we remark that many interesting actions of  $SL(n, \mathbf{R})$  on spheres, for  $n \geq 3$ , can be found in the papers of Fuichi Uchida (see for example [46, 47, 48]).

# 8. A $C^{\infty}$ -ACTION OF $SL(2, \mathbf{R})$ WHICH IS NOT LINEARIZABLE

Here we give a variation of the Guillemin-Sternberg example a  $C^{\infty}$ -action of the Lie algebra  $\mathfrak{sl}(2,\mathbf{R})$  on  $\mathbf{R}^3$  which is not linearizable. The action we give below integrates to a  $C^{\infty}$  non-linearizable  $SL(2,\mathbf{R})$ -action. It is obtained by deforming the adjoint action of  $SL(2,\mathbf{R})$  on its Lie algebra. The constructed action is clearly non-linearizable since it has an orbit of dimension 3.

By differentiation, the adjoint action of  $SL(2, \mathbf{R})$  defines a Lie algebra  $\mathfrak{g}$  (isomorphic to  $\mathfrak{sl}(2, \mathbf{R})$ ) of vector fields on  $\mathbf{R}^3$ . This algebra can be explicitly computed as follows: choose an element  $h \in \mathfrak{sl}(2, \mathbf{R})$ , take its exponential  $\exp h$ , and compute the derivative of the adjoint map  $Ad(\exp(th))$  at t = 0. A convenient basis for  $\mathfrak{g}$  is:

$$X = z \frac{\partial}{\partial y} + y \frac{\partial}{\partial z}, \quad Y = z \frac{\partial}{\partial x} + x \frac{\partial}{\partial z}, \quad R = x \frac{\partial}{\partial y} - y \frac{\partial}{\partial x}.$$

Here R is the derivative of  $Ad(\exp(th))$  where  $h = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ . The commutator relations are:

$$[X, Y] = -R, \quad [R, X] = Y. \quad [R, Y] = -X.$$