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Autor: CAIRNS, Grant / Ghys, Étienne
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4. $SL(n, \mathbf{R})$ -ACTIONS ON \mathbf{R}^n FOR $n \geq 3$

Let $n \geq 3$. We first give examples of C^0 -actions of $SL(n, \mathbf{R})$ on \mathbf{R}^n . Consider the canonical projective action of $SL(n, \mathbf{R})$ on S^{n-1} . Let Δ^+ be the radial half-line through the first basis element e_1 and let H denote the subgroup of $SL(n, \mathbf{R})$ that fixes Δ^+ . So $SL(n, \mathbf{R})/H \cong S^{n-1}$. Consider the homomorphism

$$\psi: (A_{ij}) \in H \mapsto \ln A_{11} \in \mathbf{R}.$$

Notice that one obtains a linear action of H on $\mathbf{R}_*^+ = (0, \infty)$ by setting $h(x) = e^{\psi(h)}x$, for all $h \in H$, $x \in \mathbf{R}_*^+$. Obviously this is conjugate to the H -action on Δ^+ . It follows from Lemma 3.7 that the action of $SL(n, \mathbf{R})$ obtained by suspension of this action of H on \mathbf{R}_*^+ is the canonical linear action of $SL(n, \mathbf{R})$ on $\mathbf{R}^n \setminus \{0\}$. In fact, the map

$$\psi: [g.x] \in (SL(n, \mathbf{R}) \times \mathbf{R}_*^+)/H \mapsto g(xe_1) \in \mathbf{R}^n \setminus \{0\}$$

is an isomorphism. We now deform the action of H . Choose a topological flow $(\phi^t)_{t \in \mathbf{R}}$ on $\mathbf{R}^+ = [0, \infty)$, fixing 0. This defines an action of H on \mathbf{R}_*^+ by setting $h(x) = \phi^{\psi(h)}(x)$, for all $h \in H$, $x \in \mathbf{R}_*^+$. Now suspend this action of H and let Φ denote the resulting action of $SL(n, \mathbf{R})$ on the space $M = (SL(n, \mathbf{R}) \times \mathbf{R}_*^+)/H$. The space M fibres over S^{n-1} , with fibre \mathbf{R}_*^+ , and the structure group is orientation preserving. So topologically, M is $\mathbf{R}_*^+ \times S^{n-1}$. Thus, identifying $S^{n-1} \times \{0\}$ to a point, we obtain an $SL(n, \mathbf{R})$ -action on \mathbf{R}^n . The fixed points of the flow ϕ correspond to orbits in \mathbf{R}^n which are spheres of dimension $n-1$. In general, an n -dimensional orbit is either all of $\mathbf{R}^n \setminus \{0\}$, as in the linear case, or it is a spherical shell, bounded by S^{n-1} orbits, or a punctured ball bounded by an S^{n-1} orbit, or the exterior of an S^{n-1} orbit. In all cases, the n -dimensional orbits are conjugate to the canonical linear one on $\mathbf{R}^n \setminus \{0\}$, by Theorem 3.5(c).

THEOREM 4.1. *For all $n \geq 3$, every non-trivial C^0 -action of $SL(n, \mathbf{R})$ on $(\mathbf{R}^n, 0)$ is conjugate to one of the above actions Φ .*

Proof. Suppose that we have a non-trivial C^0 -action of $SL(n, \mathbf{R})$ on $(\mathbf{R}^n, 0)$. First use Proposition 3.8 to linearize the $SO(n)$ -action. Then by Lemma 3.9, the $SL(n, \mathbf{R})$ -action preserves the radial lines. Hence the radial projection $\mathbf{R}^n \setminus \{0\} \rightarrow S^{n-1}$ is equivariant, where the action of $SL(n, \mathbf{R})$ on S^{n-1} is the canonical projective one. Let H be the stabilizer of the radial half-line Δ^+ through e_1 , as above. So the action of $SL(n, \mathbf{R})$ on $\mathbf{R}^n \setminus \{0\}$ is induced by some action of H on \mathbf{R} . Notice that this action is trivial when restricted to

$SO(n-1)$. It remains to consider all actions of H on \mathbf{R} which are trivial on $SO(n-1)$. Again, by Lie [23, *ibid.*], these are given by homomorphisms from H to \mathbf{R} , \mathbf{Aff} , or (some cover of) $PSL(2, \mathbf{R})$. We have the homomorphism $\psi: (A_{ij}) \in H \mapsto \ln A_{11} \in \mathbf{R}$. Note that $\ker \psi = SL(n-1, \mathbf{R}) \ltimes \mathbf{R}^{n-1}$. But it is easy to see that there are no non-trivial homomorphisms of $\ker \psi$ to \mathbf{R} or \mathbf{Aff} . There are no non-trivial homomorphisms of $\ker \psi$ to $SL(2, \mathbf{R})$, except in the case $n=3$, and in this case there are no such homomorphisms which are trivial on $SO(n-1)$. So the only possibility left is that H acts on \mathbf{R} by some flow. Finally, we put back the origin, as in the proof of Proposition 3.8. This completes the proof of the theorem. \square

We now prove Theorem 1.1 for $n \geq 3$.

THEOREM 4.2. *For all $n \geq 3$ and $k = 1, \dots, \infty$, every C^k -action of $SL(n, \mathbf{R})$ on $(\mathbf{R}^n, 0)$ is C^k -linearizable.*

Proof. Let $n \geq 3$ and $k = 1, \dots, \infty$ and suppose that we have a non-trivial C^k -action of $SL(n, \mathbf{R})$ on $(\mathbf{R}^n, 0)$. By Remark 3.4, we may assume that the differential of the action at the origin is either the identity or the map $g \mapsto (g^{-1})^t$. We will assume that it is the identity; the other possibility can be handled using the same argument.

Linearizing the $SO(n)$ -action, using the Bochner-Cartan theorem, one may assume that the $SO(n)$ -action is the canonical one. Then by Lemma 3.9, the $SL(n, \mathbf{R})$ -action preserves the radial lines. Let Δ denote the radial line through the first of the canonical basis elements, e_1 . Consider $H = \text{Stab}_{SL(n, \mathbf{R})}(\Delta)$, as before. So, as we saw in the proof of Theorem 4.1, H defines a C^k -flow on Δ . This flow is hyperbolic, by the first paragraph. Hence by Theorem 2.5, this flow is linearizable by some local C^k -diffeomorphism f of $\Delta(\cong \mathbf{R})$. So, after conjugacy, we may assume that H acts linearly on Δ . Now define the local C^k -diffeomorphism F of \mathbf{R}^n by the formula:

$$(2) \quad F(x) = \begin{cases} \frac{f(\|x\|)}{\|x\|} x, & x \neq 0 \\ 0, & x = 0. \end{cases}$$

To see that F is of class C^k , the key point is to verify that f is a C^k odd function on \mathbf{R} . This follows easily from the fact that the flow on Δ commutes with $\text{Stab}_{SO(n, \mathbf{R})}(\Delta)$, and the $SO(n)$ -action is linear.

Now notice that F agrees with f on $\Delta^+ = \{te_1 \in \Delta : t \geq 0\}$, and as F commutes with the $SO(n)$ -action, the $SO(n)$ -action is unchanged by conjugation by F . In particular, the $SO(n)$ -action still commutes with dilations.

It follows that after conjugation by F , the $SL(n, \mathbf{R})$ -action commutes with dilations. Indeed, consider the conjugated $SL(n, \mathbf{R})$ -action. If $f \in SL(n, \mathbf{R})$, $x \in \mathbf{R}^n$ and $\lambda > 0$, then choose $a, b \in SO(n)$ such that $ax \in \Delta^+$ and $bf(\lambda x) \in \Delta^+$. Provided x is sufficiently close to 0, ax and $bf(\lambda x)$ will lie in the domain of f . Then $bfa^{-1} \in H$ and so

$$\begin{aligned} f(\lambda x) &= b^{-1}bfa^{-1}a(\lambda x) = b^{-1}(bfg^{-1})\lambda a(x) \\ &= b^{-1}\lambda(bfa^{-1})a(x) = \lambda b^{-1}(bfa^{-1})a(x) \\ &= \lambda f(x). \end{aligned}$$

The proof of the theorem is then completed by the following well known result (cf. [17, Lemma 2.1.4]). \square

LEMMA 4.3. *Every C^1 map commuting with dilations is linear.*

Proof. Suppose that f is a C^1 -diffeomorphism of \mathbf{R}^n which commutes with dilations. By comparing the differential of $\lambda \cdot f$ and $f \circ \lambda$ at x we have $\lambda df|_x = \lambda df|_{\lambda x}$, for each $\lambda > 0$ and every $x \in \mathbf{R}^n$. Hence $df|_x = df|_{\lambda x}$ and so df is constant on the radial lines. Thus $df|_x = df|_0$ for all x and so f is linear. \square

5. THE ADJOINT REPRESENTATION OF $SL(2, \mathbf{R})$

Let us recall some facts concerning the linear representations of $SL(2, \mathbf{R})$. Let $P_l(\mathbf{R}^2)$ denote the space of real valued homogeneous polynomials, of two variables, of degree l . As a vector space, $P_l(\mathbf{R}^2) \cong \mathbf{R}^{l+1}$, and the action of $SL(2, \mathbf{R})$ on \mathbf{R}^2 defines a linear action on $P_l(\mathbf{R}^2)$: up to isomorphism, this is the (unique) irreducible representation of $SL(2, \mathbf{R})$ in dimension $l + 1$. In dimension 3, there is another useful realization of the polynomial representation, called the *adjoint representation*. Notice that the group $SL(2, \mathbf{R})$ acts by the adjoint representation on its Lie algebra $\mathfrak{sl}(2, \mathbf{R})$. Of course, $\mathfrak{sl}(2, \mathbf{R})$ is the space of 2×2 real traceless matrices; so as a vector space, $\mathfrak{sl}(2, \mathbf{R}) \cong \mathbf{R}^3$. The adjoint representation $Ad: SL(2, \mathbf{R}) \rightarrow GL(3, \mathbf{R})$, defined by

$$Ad(g): h \mapsto ghg^{-1}, \quad \forall g \in SL(2, \mathbf{R}), \quad h \in \mathfrak{sl}(2, \mathbf{R}),$$

is an irreducible linear representation. In fact, an explicit equivariant isomorphism $\psi: \mathfrak{sl}(2, \mathbf{R}) \rightarrow P_2(\mathbf{R}^2)$ is obtained by taking $\psi(h)$, as a function of variables x and y , to be the area of the parallelogram spanned by (x, y) and $h(x, y)$. That is,