Zeitschrift: L'Enseignement Mathématique

Herausgeber: Commission Internationale de l'Enseignement Mathématique

Band: 43 (1997)

Heft: 1-2: L'ENSEIGNEMENT MATHÉMATIQUE

Artikel: THE LOCAL LINEARIZATION PROBLEM FOR SMOOTH SL(n) - ACTIONS

Autor: CAIRNS, Grant / Ghys, Étienne

Kapitel: 4. SL(n,R)-ACTIONS ON R^n FOR n 3

DOI: https://doi.org/10.5169/seals-63275

4. $SL(n,\mathbf{R})$-ACTIONS ON $\mathbf{R}^n$ FOR $n \geq 3$

Let $n \geq 3$. We first give examples of C^0-actions of $SL(n,\mathbf{R})$ on $\mathbf{R}^n$. Consider the canonical projective action of $SL(n,\mathbf{R})$ on S^{n-1}. Let Δ^+ be the radial half-line through the first basis element e_1 and let H denote the subgroup of $SL(n,\mathbf{R})$ that fixes Δ^+. So $SL(n,\mathbf{R})/H \cong S^{n-1}$. Consider the homomorphism

$$\psi\colon (A_{ij}) \in H \mapsto \ln A_{11} \in \mathbf{R}.$$

Notice that one obtains a linear action of H on $\mathbf{R}_*^+ = (0,\infty)$ by setting $h(x) = e^{\psi(h)}x$, for all $h \in H$, $x \in \mathbf{R}_*^+$. Obviously this is conjugate to the H-action on Δ^+. It follows from Lemma 3.7 that the action of $SL(n,\mathbf{R})$ obtained by suspension of this action of H on $\mathbf{R}_*^+$ is the canonical linear action of $SL(n,\mathbf{R})$ on $\mathbf{R}^n\backslash\{0\}$. In fact, the map

$$\psi\colon [g,x] \in \big(SL(n,\mathbf{R}) \times \mathbf{R}_*^+\big)/H \mapsto g(xe_1) \in \mathbf{R}^n\backslash\{0\}$$

is an isomorphism. We now deform the action of H. Choose a topological flow $(\varphi^t)_{t\in\mathbf{R}}$ on $\mathbf{R}^+ = [0,\infty)$, fixing 0. This defines an action of H on $\mathbf{R}_*^+$ by setting $h(x) = \varphi^{\psi(h)}(x)$, for all $h \in H$, $x \in \mathbf{R}_*^+$. Now suspend this action of H and let Φ denote the resulting action of $SL(n,\mathbf{R})$ on the space $M = \big(SL(n,\mathbf{R}) \times \mathbf{R}_*^+\big)/H$. The space M fibres over S^{n-1}, with fibre $\mathbf{R}_*^+$, and the structure group is orientation preserving. So topologically, M is $\mathbf{R}_*^+ \times S^{n-1}$. Thus, identifying $S^{n-1} \times \{0\}$ to a point, we obtain an $SL(n,\mathbf{R})$-action on $\mathbf{R}^n$. The fixed points of the flow φ correspond to orbits in $\mathbf{R}^n$ which are spheres of dimension $n-1$. In general, an n-dimensional orbit is either all of $\mathbf{R}^n\backslash\{0\}$, as in the linear case, or it is a spherical shell, bounded by S^{n-1} orbits, or a punctured ball bounded by an S^{n-1} orbit, or the exterior of an S^{n-1} orbit. In all cases, the n-dimensional orbits are conjugate to the canonical linear one on $\mathbf{R}^n\backslash\{0\}$, by Theorem 3.5(c).

THEOREM 4.1. *For all $n \geq 3$, every non-trivial C^0-action of $SL(n,\mathbf{R})$ on $(\mathbf{R}^n, 0)$ is conjugate to one of the above actions Φ.*

Proof. Suppose that we have a non-trivial C^0-action of $SL(n,\mathbf{R})$ on $(\mathbf{R}^n, 0)$. First use Proposition 3.8 to linearize the $SO(n)$-action. Then by Lemma 3.9, the $SL(n,\mathbf{R})$-action preserves the radial lines. Hence the radial projection $\mathbf{R}^n\backslash\{0\} \to S^{n-1}$ is equivariant, where the action of $SL(n,\mathbf{R})$ on S^{n-1} is the canonical projective one. Let H be the stabilizer of the radial half-line Δ^+ through e_1, as above. So the action of $SL(n,\mathbf{R})$ on $\mathbf{R}^n\backslash\{0\}$ is induced by some action of H on $\mathbf{R}$. Notice that this action is trivial when restricted to

$SO(n-1)$. It remains to consider all actions of H on $\mathbf{R}$ which are trivial on $SO(n-1)$. Again, by Lie [23, ibid.], these are given by homomorphisms from H to $\mathbf{R}$, $\mathbf{Aff}$, or (some cover of) $PSL(2,\mathbf{R})$. We have the homomorphism $\psi\colon (A_{ij}) \in H \mapsto \ln A_{11} \in \mathbf{R}$. Note that $\ker\psi = SL(n-1,\mathbf{R}) \ltimes \mathbf{R}^{n-1}$. But it is easy to see that there are no non-trivial homomorphisms of $\ker\psi$ to $\mathbf{R}$ or $\mathbf{Aff}$. There are no non-trivial homomorphisms of $\ker\psi$ to $SL(2,\mathbf{R})$, except in the case $n=3$, and in this case there are no such homomorphisms which are trivial on $SO(n-1)$. So the only possibility left is that H acts on $\mathbf{R}$ by some flow. Finally, we put back the origin, as in the proof of Proposition 3.8. This completes the proof of the theorem. $\square$

We now prove Theorem 1.1 for $n \geq 3$.

THEOREM 4.2. *For all $n \geq 3$ and $k = 1, \ldots, \infty$, every C^k-action of $SL(n,\mathbf{R})$ on $(\mathbf{R}^n, 0)$ is C^k-linearizable.*

Proof. Let $n \geq 3$ and $k = 1, \ldots, \infty$ and suppose that we have a non-trivial C^k-action of $SL(n,\mathbf{R})$ on $(\mathbf{R}^n, 0)$. By Remark 3.4, we may assume that the differential of the action at the origin is either the identity or the map $g \mapsto (g^{-1})^t$. We will assume that it is the identity; the other possibility can be handled using the same argument.

Linearizing the $SO(n)$-action, using the Bochner-Cartan theorem, one may assume that the $SO(n)$-action is the canonical one. Then by Lemma 3.9, the $SL(n,\mathbf{R})$-action preserves the radial lines. Let Δ denote the radial line through the first of the canonical basis elements, e_1. Consider $H = \mathrm{Stab}_{SL(n,\mathbf{R})}(\Delta)$, as before. So, as we saw in the proof of Theorem 4.1, H defines a C^k-flow on Δ. This flow is hyperbolic, by the first paragraph. Hence by Theorem 2.5, this flow is linearizable by some local C^k-diffeomorphism f of $\Delta(\cong \mathbf{R})$. So, after conjugacy, we may assume that H acts linearly on Δ. Now define the local C^k-diffeomorphism F of $\mathbf{R}^n$ by the formula:

$$F(x) = \begin{cases} \frac{f(\|x\|)}{\|x\|}\, x, & x \neq 0 \\ 0, & x = 0 . \end{cases} \tag{2}$$

To see that F is of class C^k, the key point is to verify that f is a C^k *odd* function on $\mathbf{R}$. This follows easily from the fact that the flow on Δ commutes with $\mathrm{Stab}_{SO(n,\mathbf{R})}(\Delta)$, and the $SO(n)$-action is linear.

Now notice that F agrees with f on $\Delta^+ = \{t e_1 \in \Delta \ : \ t \geq 0\}$, and as F commutes with the $SO(n)$-action, the $SO(n)$-action is unchanged by conjugation by F. In particular, the $SO(n)$-action still commutes with dilations.

It follows that after conjugation by F, the $SL(n, \mathbf{R})$-action commutes with dilations. Indeed, consider the conjugated $SL(n, \mathbf{R})$-action. If $f \in SL(n, \mathbf{R})$, $x \in \mathbf{R}^n$ and $\lambda > 0$, then choose $a, b \in SO(n)$ such that $ax \in \Delta^+$ and $bf(\lambda x) \in \Delta^+$. Provided x is sufficiently close to 0, ax and $bf(\lambda x)$ will lie in the domain of f. Then $bfa^{-1} \in H$ and so

$$\begin{aligned} f(\lambda x) &= b^{-1} bfa^{-1} a(\lambda x) = b^{-1}(bfg^{-1})\, \lambda a(x) \\ &= b^{-1} \lambda (bfa^{-1})\, a(x) = \lambda b^{-1}(bfa^{-1})\, a(x) \\ &= \lambda f(x)\,. \end{aligned}$$

The proof of the theorem is then completed by the following well known result (cf. [17, Lemma 2.1.4]). □

LEMMA 4.3. *Every C^1 map commuting with dilations is linear.*

Proof. Suppose that f is a C^1-diffeomorphism of $\mathbf{R}^n$ which commutes with dilations. By comparing the differential of $\lambda . f$ and $f \circ \lambda$ at x we have $\lambda df|_x = \lambda df|_{\lambda x}$, for each $\lambda > 0$ and every $x \in \mathbf{R}^n$. Hence $df|_x = df|_{\lambda x}$ and so df is constant on the radial lines. Thus $df|_x = df|_0$ for all x and so f is linear. □

5. THE ADJOINT REPRESENTATION OF $SL(2, \mathbf{R})$

Let us recall some facts concerning the linear representations of $SL(2, \mathbf{R})$. Let $P_l(\mathbf{R}^2)$ denote the space of real valued homogeneous polynomials, of two variables, of degree l. As a vector space, $P_l(\mathbf{R}^2) \cong \mathbf{R}^{l+1}$, and the action of $SL(2, \mathbf{R})$ on $\mathbf{R}^2$ defines a linear action on $P_l(\mathbf{R}^2)$: up to isomorphism, this is the (unique) irreducible representation of $SL(2, \mathbf{R})$ in dimension $l+1$. In dimension 3, there is another useful realization of the polynomial representation, called the *adjoint representation*. Notice that the group $SL(2, \mathbf{R})$ acts by the adjoint representation on its Lie algebra $\mathfrak{sl}(2, \mathbf{R})$. Of course, $\mathfrak{sl}(2, \mathbf{R})$ is the space of 2×2 real traceless matrices; so as a vector space, $\mathfrak{sl}(2, \mathbf{R}) \cong \mathbf{R}^3$. The adjoint representation $Ad\colon SL(2, \mathbf{R}) \to GL(3, \mathbf{R})$, defined by

$$Ad(g)\colon h \mapsto ghg^{-1}, \quad \forall g \in SL(2, \mathbf{R}),\ h \in \mathfrak{sl}(2, \mathbf{R}),$$

is an irreducible linear representation. In fact, an explicit equivariant isomorphism $\psi\colon \mathfrak{sl}(2, \mathbf{R}) \to P_2(\mathbf{R}^2)$ is obtained by taking $\psi(h)$, as a function of variables x and y, to be the area of the parallelogram spanned by (x, y) and $h(x, y)$. That is,