

# 3. Preparatory results

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So, setting  $h_l = \eta h_{l-1}$ , we have that  $T^l(h_l g h_l^{-1}) = D(g)$ , for every  $g \in SL(n, \mathbf{R})$ . By induction, we have elements  $h_l \in \widehat{\text{Diff}}(\mathbf{R}^m, 0)$  such that  $T^l(h_l g h_l^{-1}) = D(g)$  for all  $l > 0$ . Finally set  $h = \lim_{l \rightarrow \infty} h_l$ . This makes sense in  $\widehat{\text{Diff}}(\mathbf{R}^m, 0)$  and by construction,  $h$  formally linearizes the action  $\Phi$ .

### 3. PREPARATORY RESULTS

First let us make some general comments:

REMARK 3.1. If a Lie group  $G$  acts on a topological manifold, then the restriction of the action to each orbit is a transitive  $G$ -action; that is, each orbit is a homogeneous space  $G/H$  for some closed subgroup  $H \subset G$ . In particular, transitive  $C^0$ -actions of  $SL(n, \mathbf{R})$  are conjugate to analytic  $SL(n, \mathbf{R})$ -actions.

REMARK 3.2. Every non-trivial continuous action of  $SL(n, \mathbf{R})$  is either faithful, or factors through a faithful action of  $PSL(n, \mathbf{R})$ . Indeed, not only is  $SL(n, \mathbf{R})$  simple as a Lie group (that is, its proper normal subgroups are discrete), but when  $n$  is odd it is simple as an abstract group and when  $n$  is even  $PSL(n, \mathbf{R}) = SL(n, \mathbf{R})/\{\pm 1\}$  is simple as an abstract group. In particular, if  $n$  is odd, every non-trivial continuous action of  $SL(n, \mathbf{R})$  is faithful. If  $n$  is even, non-faithful  $SL(n, \mathbf{R})$ -actions are common: see, for example, the adjoint action of  $SL(n, \mathbf{R})$  for  $n$  even, or the irreducible  $SL(2, \mathbf{R})$ -representation on  $\mathbf{R}^{2p+1}$  (see Section 5).

REMARK 3.3. Every non-trivial  $C^1$ -action of  $SL(n, \mathbf{R})$  on  $(\mathbf{R}^n, 0)$  is faithful. Indeed, the differential at the origin defines a homomorphism  $D: SL(n, \mathbf{R}) \rightarrow GL(n, \mathbf{R})$ . In fact, since  $SL(n, \mathbf{R})$  is a simple Lie group, the image of  $D$  is contained in  $SL(n, \mathbf{R})$ . By Thurston's stability theorem,  $D$  can't be trivial. So, for dimension reasons,  $D$  maps onto  $SL(n, \mathbf{R})$ . If an  $SL(n, \mathbf{R})$ -action is not faithful, then by the previous Remark,  $n$  is even and the element  $-1$  acts trivially. But then  $D$  defines a homomorphism from  $PSL(n, \mathbf{R})$  onto  $SL(n, \mathbf{R})$ , which is impossible since  $PSL(n, \mathbf{R})$  is simple.

REMARK 3.4. Suppose one has a  $C^1$ -action of  $SL(n, \mathbf{R})$  on  $(\mathbf{R}^n, 0)$ . By the previous Remark, the differential  $D$  defines an automorphism of  $SL(n, \mathbf{R})$ . Let  $\sigma$  be the automorphism of  $SL(n, \mathbf{R})$  defined by  $\sigma(g) = (g^{-1})^t$ , and let  $\tau$  the automorphism given by conjugation by the matrix

$$\begin{pmatrix} -1 & 0 \\ 0 & \text{Id}_{n-1} \end{pmatrix} \in GL(n, \mathbf{R}).$$

Recall (see [16, Theorem IX.5]) that the group of outer automorphisms of  $SL(n, \mathbf{R})$  is generated by the involution  $\sigma$  if  $n$  is odd, and it is the group of order 4 generated by  $\sigma$  and  $\tau$  if  $n$  is even — except when  $n = 2$ , in which case  $\sigma$  is the inner automorphism generated by conjugation by  $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ . Hence, up to conjugacy by an element of  $GL(n, \mathbf{R})$ , we may assume that the differential  $D$  is either the identity or the map  $\sigma$ .

Part (a) of the following theorem is classical (see [30, Chap. VI, Theorem 2]). Parts (b) and (c) could be deduced from Dynkin's classification of maximal subgroups of semi-simple Lie groups [8]; we give a more direct proof. We treat the case  $n = 2$  of Part (c) in Section 6 below.

THEOREM 3.5.

- (a) *There is no non-trivial  $C^0$ -action of  $SL(n, \mathbf{R})$  on any topological manifold of dimension  $m < n - 1$ .*
- (b) *Every non-trivial  $C^0$ -action of  $SL(n, \mathbf{R})$  on an  $(n - 1)$ -dimensional connected topological manifold is transitive and is conjugate to the projective action of  $SL(n, \mathbf{R})$  on either  $S^{n-1}$  or  $\mathbf{R}P^{n-1}$ .*
- (c) *For  $n \geq 3$ , every transitive  $C^0$ -action of  $SL(n, \mathbf{R})$  on a non-compact  $n$ -dimensional topological manifold is conjugate, after possibly pre-composing with some automorphism of  $SL(n, \mathbf{R})$ , to the canonical action of  $SL(n, \mathbf{R})$  on  $\mathbf{R}^n \setminus \{0\}$  or  $(\mathbf{R}^n \setminus \{0\}) / \{\pm \text{Id}\} \cong \mathbf{R}P^{n-1} \times \mathbf{R}$ .*

*Proof.* (a) Suppose that  $H$  is a closed subgroup of  $SL(n, \mathbf{R})$  of codimension  $m$ . Consider the restricted  $SO(n)$ -action. Choose any Riemannian metric on the smooth manifold  $M = SL(n, \mathbf{R})/H$  and average it by the  $SO(n)$ -action. Then  $SO(n)$  acts isometrically, for the averaged metric. But the group of isometries of  $M$  has dimension at most  $m(m + 1)/2$ , by [19, Theorem II.3.1]. So

$$\dim SO(n) = \binom{n}{2} \leq \binom{m + 1}{2}.$$

Hence  $n \leq m + 1$ , as required.

(b) Suppose one has a non-trivial  $C^0$ -action of  $SL(n, \mathbf{R})$  on an  $(n - 1)$ -dimensional connected topological manifold  $M$ . By (a), this action is transitive and  $M = G/H$  for some closed subgroup  $H \subset G$ . Then the restricted  $SO(n)$ -action gives a compact group of isometries of  $M$  of dimension  $n(n - 1)/2$ . It follows from [19, Theorem II.3.1] that  $M$  is the round sphere  $S^{n-1}$ , or projective space  $\mathbf{R}P^{n-1}$ , and the action is the canonical one.

(c) Consider a transitive  $C^0$ -action of  $SL(n, \mathbf{R})$  on an  $n$ -dimensional topological manifold  $M$  and let  $H$  denote the stabilizer of some point so that  $M$  can be identified with the homogeneous space  $SL(n, \mathbf{R})/H$ . We first deal with the case where  $H$  is connected, since the other cases can be reduced to this by taking a covering of the corresponding homogeneous space. We begin by showing that the linear action of  $H \subset SL(n, \mathbf{R})$  on  $\mathbf{R}^n$  is reducible and fixes a line or a hyperplane.

Suppose first by contradiction that the complexified representation of the Lie algebra  $\mathfrak{h} \otimes \mathbf{C} \subset \mathfrak{sl}(n, \mathbf{C})$  is irreducible, where  $\mathfrak{h}$  denotes the Lie algebra of  $H$ . By a well known theorem of Lie, the radical of  $\mathfrak{h} \otimes \mathbf{C}$  preserves some line in  $\mathbf{C}^n$  and since we assume that  $\mathfrak{h} \otimes \mathbf{C}$  is irreducible, the only possibility is that this radical is Abelian and acts by homotheties. In other words,  $\mathfrak{h} \otimes \mathbf{C}$  is a reductive algebra. By taking suitable real forms, one would have a compact subgroup  $K$  in  $SU(n)$  whose real codimension is  $n$ . Now, as before, one can consider  $SU(n)$  as a group of isometries of the  $n$ -dimensional manifold  $SU(n)/K$ . This would imply that  $\dim SU(n) = n^2 - 1 \leq n(n - 1)/2$  which is a contradiction.

On the other hand, if  $\mathfrak{h} \otimes \mathbf{C} \subset \mathfrak{sl}(n, \mathbf{C})$  is a reducible representation, then  $\mathfrak{h} \otimes \mathbf{C} \subset \mathfrak{sl}(n, \mathbf{C})$  is contained (up to conjugacy) in the algebra of matrices preserving  $\mathbf{C}^p \times \{0\}$  (for some  $0 < p < n$ ) which is of codimension  $p(n - p)$ . Therefore  $p(n - p) \leq n$  so that  $p = 1$  or  $n - 1$ . This means that there is a complex line or a complex hyperplane fixed by  $\mathfrak{h} \otimes \mathbf{C}$ . This line or hyperplane has to be invariant under complex conjugation; otherwise we would have an invariant complex subspace of dimension or codimension 2 and this is not possible since  $H$  has codimension exactly  $n$ . It follows that  $H$  fixes a line or a hyperplane.

If  $H$  fixes a hyperplane, replace it by  $\sigma(H)$  where  $\sigma$  is the automorphism of  $SL(n, \mathbf{R})$  defined by  $\sigma(g) = (g^{-1})^t$ . This amounts to changing the action of  $SL(n, \mathbf{R})$  under consideration by pre-composing with  $\sigma$ . So we can assume that  $H$  is contained in the stabilizer  $H'$  of the radial half-line  $\Delta^+$  through the first vector  $e_1$  of the canonical basis in  $\mathbf{R}^n$ . Moreover,  $H$  is a codimension one subgroup of  $H'$ .

By Lie [23] (see also [33, Part II, Chap. 6, Theorem 2.1]), the connected codimension one closed subgroups of  $H'$  are given by homomorphisms  $\psi$  from  $H'$  to  $\mathbf{R}$ ,  $\mathbf{Aff}$ , or (some cover of)  $PSL(2, \mathbf{R})$ , where

$$\mathbf{Aff} = \left\{ \begin{pmatrix} a & b \\ 0 & 1/a \end{pmatrix} : a > 0 \right\}$$

is the group of affine transformations of the line. More precisely,  $H$  is (the component of the identity of) the inverse image by  $\psi$  of a codimension one subgroup, which is trivial in the case of  $\mathbf{R}$ , the subgroup of homotheties ( $b = 0$ ) in the case of  $\mathbf{Aff}$  and the upper triangular subgroup in the case of  $PSL(2, \mathbf{R})$ . It is easy to see that there are no non-trivial homomorphisms of  $H'$  to  $\mathbf{Aff}$ . There are no non-trivial homomorphisms of  $H'$  to (any cover of)  $PSL(2, \mathbf{R})$ , except in the case  $n = 3$ . In this special case  $n = 3$ , one finds that  $H$  is the restricted upper-triangular group

$$U = \left\{ \begin{pmatrix} a & b & c \\ 0 & d & e \\ 0 & 0 & f \end{pmatrix} : a > 0 \right\},$$

which gives the compact flag manifold  $SL(3, \mathbf{R})/U \cong S^3$ . Finally, up to a multiplicative constant, there is a unique homomorphism from  $H'$  to  $\mathbf{R}$ :

$$\psi: (A_{ij}) \in H' \mapsto \ln A_{11} \in \mathbf{R}.$$

Note that here  $H = \ker \psi$  is precisely the stabilizer  $\text{Stab}_{SL(n, \mathbf{R})}(e_1)$  of  $e_1$  so that here  $SL(n, \mathbf{R})/H$  is the homogeneous space  $\mathbf{R}^n \setminus \{0\}$ .

Thus we have dealt with the case where  $H$  is connected. Suppose that  $H$  is not connected, and let  $H_0$  be its connected component of the identity. Now  $H_0$  is a normal subgroup of  $H$ , and from above, by conjugation we may take  $H_0$  to be either the group  $\text{Stab}_{SL(n, \mathbf{R})}(e_1)$ , or the group  $U$ . If  $H_0 = \text{Stab}_{SL(n, \mathbf{R})}(e_1)$ , notice that the normalizer of  $H_0$  is the stabilizer  $H'$  of the radial half-line  $\Delta^+$ . It follows that  $H/H_0$  is a discrete subgroup of  $\mathbf{R}$ . If  $H/H_0$  is finite, then  $H/H_0 = \pm 1$  and so the quotient space is  $\mathbf{R}^n \setminus \{0\} / \{\pm \text{Id}\}$ . If  $H/H_0$  is infinite, then it is either infinite cyclic, or infinite cyclic cross  $\mathbf{Z}/2\mathbf{Z}$ , and in either case the quotient space is compact. If  $H_0 = U$ , the normalizer of  $H_0$  is the full group  $\bar{U}$  of upper-triangular matrices: there are 3 possibilities here, but in each case we get a compact quotient space.

This completes the proof of the theorem.  $\square$

We now describe a useful method of extending an action of a subgroup to an action of the larger group. This method is very general and variations of it

appear in various branches of mathematics: “induced module” in representation theory, “suspension” in dynamical systems, etc. In particular, it was used in an essential way in Schneider’s classification of analytic  $SL(2, \mathbf{R})$ -actions on surfaces [37]. Suppose that  $H$  is a closed subgroup of a Lie group  $G$  and suppose that  $H$  acts continuously on a topological space  $F$ . So  $H$  acts diagonally on  $G \times F$ , where  $g \in H \subset G$  acts on the first factor by right translation by  $g^{-1}$ . Let  $E = (G \times F)/H$  denote the quotient space. So  $E$  fibres over the space  $G/H$  of left cosets of  $H$ , with fibre  $F$ . Now notice that  $G$  acts on  $G \times F$  by left translation on the first factor, and this defines an action of  $G$  on  $E$ .

**DEFINITION 3.6.** The action of  $G$  on  $E$  just described is called the *suspension of the action of  $H$  on  $F$* .

Notice that for such an action, there is a  $H$ -invariant subspace  $F'$  in  $E$ , which is  $H$ -equivariantly homeomorphic to  $F$ , and which has the property that  $\text{Stab}_H(x) = \text{Stab}_G(x)$ , for all  $x \in F'$ . Indeed, one can take  $F' = \pi^{-1}(H)$ , where  $\pi: E \rightarrow G/H$  is the natural fibration. Given  $f \in F$  and  $g \in G$ , let  $[g, f]$  denote the image in  $E$  of  $(g, f)$  under the quotient map  $G \times F \rightarrow E$ . Then  $\pi[g, f] = gH$ , and  $F' = \{[1, f] : f \in F\} (SL(n, \mathbf{R}))$ .

Conversely, one has:

**LEMMA 3.7.** *Let  $H$  be a closed subgroup of a Lie group  $G$ . Suppose that  $G$  acts continuously on a topological space  $M$  and that there is a  $G$ -equivariant fibration  $p: M \rightarrow G/H$ . Then the  $G$ -action on  $M$  is conjugate to the suspension of the action of  $H$  on the fibre  $F = p^{-1}(H)$ . More precisely, if  $E = (G \times F)/H$ , then there is a  $G$ -equivariant homeomorphism from  $M$  to  $E$  which projects to the identity map on  $G/H$ .*

*Proof.* We define a function  $\psi: M \rightarrow E$  as follows: for each  $x \in M$  we set

$$\psi(x) = [g, g^{-1}(x)],$$

where  $p(x) = gH$ . Note that this makes sense since  $g^{-1}(x) \in F$  and the definition of  $\psi(x)$  doesn’t depend upon the choice of  $g$ . By construction,  $\psi$  is  $G$ -equivariant and projects to the identity map on  $G/H$ . Finally, it is easy to see that  $\psi$  is a homeomorphism.  $\square$

By Remark 2.2,  $SO(n)$ -actions of class  $C^0$  on  $(\mathbf{R}^m, 0)$  are not always linearizable. Despite this, we have the following result, which was proved for the cases  $n \leq 3$  in [30, Chapter VI.6.5] and was conjectured therein for all  $n$ .

PROPOSITION 3.8. *Every faithful  $C^0$ -action of  $SO(n)$  on  $(\mathbf{R}^n, 0)$  is globally conjugate to the canonical linear action.*

*Proof.* By the proof of Theorem 3.5(a), the orbits of the  $SO(n)$ -action have dimension  $\geq n - 1$ . In fact, there cannot be any  $SO(n)$ -orbit of dimension  $n$ , since otherwise it would be all of  $\mathbf{R}^n \setminus \{0\}$ , which is impossible, by the compactness of  $SO(n)$ . By the proof of Theorem 3.5(b), the only  $SO(n)$ -orbits of dimension  $n - 1$  are  $S^{n-1}$  and  $\mathbf{R}P^{n-1}$ , and the actions on them are conjugate to the canonical projective ones. In fact, for  $n \geq 3$  there can be no orbit homeomorphic to  $\mathbf{R}P^{n-1}$ , because  $\mathbf{R}P^{n-1}$  does not embed in  $\mathbf{R}^n$  [6, Theorem 10.12]. So each orbit of  $SO(n)$  is a  $(n - 1)$ -dimensional sphere or a fixed point. It then follows from [30, *ibid.*] that 0 is the unique fixed point and there is a continuous ray  $\gamma$  beginning at 0 which meets each  $SO(n)$ -orbit exactly once.

First consider the  $n = 2$  case. Note that the  $SO(2)$ -action on  $\mathbf{R}^2 \setminus \{0\}$  is free. Indeed, let  $g \in SO(2)$  and suppose that  $x \in \mathbf{R}^2 \setminus \{0\}$  belongs to the fixed point set  $\text{Fix}(g)$  of the action of  $g$  on  $\mathbf{R}^2$ . Then  $\text{Fix}(g)$  contains 0 as well as the entire orbit of  $x$  by  $SO(2)$ . By Eilenberg's theorem [9], since  $g$  is orientation preserving, the action of  $g$  on  $\mathbf{R}^2$  is topologically conjugate to a rotation. So, as  $g$  has more than one fixed point, we must have  $\text{Fix}(g) = \mathbf{R}^2$ . Hence, as the  $SO(2)$ -action on  $\mathbf{R}^2$  is faithful by hypothesis, we have  $g = \text{Id}$ , as claimed. Now define the map  $\phi: \mathbf{R}^2 \rightarrow \mathbf{R}^2$  by setting

$$\phi(h\gamma(t)) = h \cdot \begin{pmatrix} t \\ 0 \end{pmatrix},$$

for all  $t \in [0, \infty)$ ,  $h \in SO(2)$ , where  $h$  acts on the left via the given  $SO(2)$ -action, and on the right by matrix multiplication. By construction,  $\phi$  conjugates the given  $SO(2)$ -action to the canonical linear action.

Now suppose  $n > 2$ . Let  $\{e_1, \dots, e_n\}$  denote the canonical basis of  $\mathbf{R}^n$ . Then, as in the proof in [30, *ibid.*], one may choose the ray  $\gamma$  to be comprised of fixed points of the restricted  $SO(n - 1)$ -action, where here  $SO(n - 1)$  is the subgroup of  $SO(n)$  which fixes the first basis vector  $e_1$ . So for each  $x \in \mathbf{R}^n$ , there is a unique number  $t \in [0, \infty)$  and an element  $g \in SO(n)$  such that  $x = g(\gamma(t))$ . Moreover, for  $x \in \mathbf{R}^n \setminus \{0\}$ , the element  $g$  is unique modulo  $SO(n - 1)$ . Consider the fibration

$$p: x \in \mathbf{R}^n \setminus \{0\} \mapsto g \in SO(n)/SO(n - 1) \cong S^{n-1}.$$

Clearly  $p$  is  $SO(n)$ -equivariant. Notice that  $p^{-1}(SO(n - 1)) = \gamma \setminus \{0\} \cong \mathbf{R}$  and the  $SO(n - 1)$ -action on this set is trivial. So, by Lemma 3.7, the action of  $SO(n)$  on  $\mathbf{R}^n \setminus \{0\}$  is conjugate to the action induced by the trivial action

of  $SO(n-1)$  on  $\mathbf{R}$ . That is, it is conjugate to the canonical action of  $SO(n)$  on  $\mathbf{R}^n \setminus \{0\}$ . It remains to put back the origin. This can obviously be done equivariantly: one merely needs to verify that it can be done continuously. However, by averaging the flat metric on  $\mathbf{R}^n$  by the original action of  $SO(n)$ , one may assume that the action is distance preserving. Thus, as  $t$  tends to 0, the  $SO(n)$ -orbits through  $\gamma(t)$  converge uniformly to 0. So the continuity of the conjugation is clear.  $\square$

We will also need the following:

LEMMA 3.9. *Let  $n \geq 3$  and suppose that one has a  $C^0$ -action of  $SL(n, \mathbf{R})$  on  $(\mathbf{R}^n, 0)$  such that the restricted action of  $SO(n)$  is the canonical linear action. Then locally the  $SL(n, \mathbf{R})$ -action preserves the radial lines.*

*Proof.* The key point is that two points of  $\mathbf{R}^n$  lie in the same radial line if and only if they have the same stabilizer under the  $SO(n)$ -action. Let  $x, y \in \mathbf{R}^n$  lie in the same radial line and let  $g \in SL(n, \mathbf{R})$ . So  $\text{Stab}_{SO(n)}(x) = \text{Stab}_{SO(n)}(y)$  and we want to show that

$$\text{Stab}_{SO(n)}(g(x)) = \text{Stab}_{SO(n)}(g(y)).$$

Since the restricted action of  $SO(n)$  is the canonical linear action, each orbit of  $SL(n, \mathbf{R})$  in  $\mathbf{R}^n \setminus \{0\}$  is either a round sphere centred at 0 or a spherical shell centred at 0. Suppose that our  $SL(n, \mathbf{R})$ -action on  $\mathbf{R}^n$  has two spherical orbits,  $S_1$  and  $S_2$  say. By Theorem 3.5(b), the  $SL(n, \mathbf{R})$ -action on each sphere is the projective one. So there is an equivariant homeomorphism  $\psi: S_1 \rightarrow S_2$ . If  $x \in S_1$  and  $y = \psi(x) \in S_2$ , we have  $g(y) = \psi(g(x))$  and as it is equivariant,  $\psi$  respects the stabilizers of the  $SO(n)$ -action. So  $\text{Stab}_{SO(n)}(g(y)) = \text{Stab}_{SO(n)}(g(x))$ , as required (and  $\psi$  is just  $\pm$  the radial projection of  $S_1$  onto  $S_2$ ).

By continuity, it remains to consider the case where  $x$  and  $y$  lie in the same open orbit of  $SL(n, \mathbf{R})$ ; that is, suppose  $y = h(x)$  for some  $h \in SL(n, \mathbf{R})$ . For all  $f \in SL(n, \mathbf{R})$ , one has  $\text{Stab}_{SO(n)}(x) = \text{Stab}_{SO(n)}(f(x))$  if and only if  $f \in \text{Norm}_{SL(n, \mathbf{R})}(\text{Stab}_{SO(n)}(x))$ . So  $h \in \text{Norm}_{SL(n, \mathbf{R})}(\text{Stab}_{SO(n)}(x))$  and we need to show that  $ghg^{-1} \in \text{Norm}_{SL(n, \mathbf{R})}(\text{Stab}_{SO(n)}(g(x)))$ . But if  $G$  is any group acting on a space  $X$  and  $H$  is a subgroup of  $G$ , then

$$\begin{aligned} g(\text{Norm}_G(\text{Stab}_H(x)))g^{-1} &= \text{Norm}_G(g(\text{Stab}_H(x)g^{-1})) \\ &= \text{Norm}_G(\text{Stab}_H(g(x))), \end{aligned}$$

for all  $x \in X$  and  $g \in G$ , as we require.