Zeitschrift: L'Enseignement Mathématique

Herausgeber: Commission Internationale de l'Enseignement Mathématique

Band: 43 (1997)

Heft: 1-2: L'ENSEIGNEMENT MATHÉMATIQUE

Artikel: THE LOCAL LINEARIZATION PROBLEM FOR SMOOTH SL(n) - ACTIONS

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Kapitel: 3. Preparatory results

DOI: https://doi.org/10.5169/seals-63275

So, setting $h_l = \eta h_{l-1}$, we have that $T^l(h_l g h_l^{-1}) = D(g)$, for every $g \in SL(n,\mathbf{R})$. By induction, we have elements $h_l \in \widehat{\mathrm{Diff}}(\mathbf{R}^m,0)$ such that $T^l(h_l g h_l^{-1}) = D(g)$ for all $l > 0$. Finally set $h = \lim_{l\to\infty} h_l$. This makes sense in $\widehat{\mathrm{Diff}}(\mathbf{R}^m,0)$ and by construction, h formally linearizes the action Φ.

3. Preparatory results

First let us make some general comments:

REMARK 3.1. If a Lie group G acts on a topological manifold, then the restriction of the action to each orbit is a transitive G-action; that is, each orbit is a homogeneous space G/H for some closed subgroup $H \subset G$. In particular, transitive C^0-actions of $SL(n,\mathbf{R})$ are conjugate to analytic $SL(n,\mathbf{R})$-actions.

REMARK 3.2. Every non-trivial continuous action of $SL(n,\mathbf{R})$ is either faithful, or factors through a faithful action of $PSL(n,\mathbf{R})$. Indeed, not only is $SL(n,\mathbf{R})$ simple as a Lie group (that is, its proper normal subgroups are discrete), but when n is odd it is simple as an abstract group and when n is even $PSL(n,\mathbf{R}) = SL(n,\mathbf{R})/\{\pm 1\}$ is simple as an abstract group. In particular, if n is odd, every non-trivial continuous action of $SL(n,\mathbf{R})$ is faithful. If n is even, non-faithful $SL(n,\mathbf{R})$-actions are common: see, for example, the adjoint action of $SL(n,\mathbf{R})$ for n even, or the irreducible $SL(2,\mathbf{R})$-representation on $\mathbf{R}^{2p+1}$ (see Section 5).

REMARK 3.3. Every non-trivial C^1-action of $SL(n,\mathbf{R})$ on $(\mathbf{R}^n,0)$ is faithful. Indeed, the differential at the origin defines a homomorphism $D\colon SL(n,\mathbf{R}) \to GL(n,\mathbf{R})$. In fact, since $SL(n,\mathbf{R})$ is a simple Lie group, the image of D is contained in $SL(n,\mathbf{R})$. By Thurston's stability theorem, D can't be trivial. So, for dimension reasons, D maps onto $SL(n,\mathbf{R})$. If an $SL(n,\mathbf{R})$-action is not faithful, then by the previous Remark, n is even and the element -1 acts trivially. But then D defines a homomorphism from $PSL(n,\mathbf{R})$ onto $SL(n,\mathbf{R})$, which is impossible since $PSL(n,\mathbf{R})$ is simple.

REMARK 3.4. Suppose one has a C^1-action of $SL(n,\mathbf{R})$ on $(\mathbf{R}^n,0)$. By the previous Remark, the differential D defines an automorphism of $SL(n,\mathbf{R})$. Let σ be the automorphism of $SL(n,\mathbf{R})$ defined by $\sigma(g)=(g^{-1})^t$, and let τ the automorphism given by conjugation by the matrix

$$\begin{pmatrix} -1 & 0 \\ 0 & \mathrm{Id}_{n-1} \end{pmatrix} \in GL(n,\mathbf{R}).$$

Recall (see [16, Theorem IX.5]) that the group of outer automorphisms of $SL(n,\mathbf{R})$ is generated by the involution σ if n is odd, and it is the group of order 4 generated by σ and τ if n is even — except when $n=2$, in which case σ is the inner automorphism generated by conjugation by $\left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$. Hence, up to conjugacy by an element of $GL(n,\mathbf{R})$, we may assume that the differential D is either the identity or the map σ.

Part (a) of the following theorem is classical (see [30, Chap. VI, Theorem 2]). Parts (b) and (c) could be deduced from Dynkin's classification of maximal subgroups of semi-simple Lie groups [8]; we give a more direct proof. We treat the case $n=2$ of Part (c) in Section 6 below.

THEOREM 3.5.

(a) There is no non-trivial C^0-action of $SL(n,\mathbf{R})$ on any topological manifold of dimension $m<n-1$.

(b) Every non-trivial C^0-action of $SL(n,\mathbf{R})$ on an $(n-1)$-dimensional connected topological manifold is transitive and is conjugate to the projective action of $SL(n,\mathbf{R})$ on either S^{n-1} or $\mathbf{R}P^{n-1}$.

(c) For $n\geq 3$, every transitive C^0-action of $SL(n,\mathbf{R})$ on a non-compact n-dimensional topological manifold is conjugate, after possibly precomposing with some automorphism of $SL(n,\mathbf{R})$, to the canonical action of $SL(n,\mathbf{R})$ on $\mathbf{R}^n\backslash\{0\}$ or $(\mathbf{R}^n\backslash\{0\})/\{\pm\mathrm{Id}\}\cong\mathbf{R}P^{n-1}\times\mathbf{R}$.

Proof. (a) Suppose that H is a closed subgroup of $SL(n,\mathbf{R})$ of codimension m. Consider the restricted $SO(n)$-action. Choose any Riemannian metric on the smooth manifold $M=SL(n,\mathbf{R})/H$ and average it by the $SO(n)$-action. Then $SO(n)$ acts isometrically, for the averaged metric. But the group of isometries of M has dimension at most $m(m+1)/2$, by [19, Theorem II.3.1]. So

$$\dim SO(n)=\binom{n}{2}\leq\binom{m+1}{2}.$$

Hence $n\leq m+1$, as required.

(b) Suppose one has a non-trivial C^0-action of $SL(n, \mathbf{R})$ on an $(n-1)$-dimensional connected topological manifold M. By (a), this action is transitive and $M = G/H$ for some closed subgroup $H \subset G$. Then the restricted $SO(n)$-action gives a compact group of isometries of M of dimension $n(n-1)/2$. It follows from [19, Theorem II.3.1] that M is the round sphere S^{n-1}, or projective space $\mathbf{R}P^{n-1}$, and the action is the canonical one.

(c) Consider a transitive C^0-action of $SL(n, \mathbf{R})$ on an n-dimensional topological manifold M and let H denote the stabilizer of some point so that M can be identified with the homogeneous space $SL(n, \mathbf{R})/H$. We first deal with the case where H is connected, since the other cases can be reduced to this by taking a covering of the corresponding homogeneous space. We begin by showing that the linear action of $H \subset SL(n, \mathbf{R})$ on $\mathbf{R}^n$ is reducible and fixes a line or a hyperplane.

Suppose first by contradiction that the complexified representation of the Lie algebra $\mathfrak{H} \otimes \mathbf{C} \subset \mathfrak{sl}(n, \mathbf{C})$ is irreducible, where $\mathfrak{H}$ denotes the Lie algebra of H. By a well known theorem of Lie, the radical of $\mathfrak{H} \otimes \mathbf{C}$ preserves some line in $\mathbf{C}^n$ and since we assume that $\mathfrak{H} \otimes \mathbf{C}$ is irreducible, the only possibility is that this radical is Abelian and acts by homotheties. In other words, $\mathfrak{H} \otimes \mathbf{C}$ is a reductive algebra. By taking suitable real forms, one would have a compact subgroup K in $SU(n)$ whose real codimension is n. Now, as before, one can consider $SU(n)$ as a group of isometries of the n-dimensional manifold $SU(n)/K$. This would imply that $\dim SU(n) = n^2 - 1 \leq n(n-1)/2$ which is a contradiction.

On the other hand, if $\mathfrak{H} \otimes \mathbf{C} \subset \mathfrak{sl}(n, \mathbf{C})$ is a reducible representation, then $\mathfrak{H} \otimes \mathbf{C} \subset \mathfrak{sl}(n, \mathbf{C})$ is contained (up to conjugacy) in the algebra of matrices preserving $\mathbf{C}^p \times \{0\}$ (for some $0 < p < n$) which is of codimension $p(n-p)$. Therefore $p(n-p) \leq n$ so that $p = 1$ or $n-1$. This means that there is a complex line or a complex hyperplane fixed by $\mathfrak{H} \otimes \mathbf{C}$. This line or hyperplane has to be invariant under complex conjugation; otherwise we would have an invariant complex subspace of dimension or codimension 2 and this is not possible since H has codimension exactly n. It follows that H fixes a line or a hyperplane.

If H fixes a hyperplane, replace it by $\sigma(H)$ where σ is the automorphism of $SL(n, \mathbf{R})$ defined by $\sigma(g) = (g^{-1})^t$. This amounts to changing the action of $SL(n, \mathbf{R})$ under consideration by pre-composing with σ. So we can assume that H is contained in the stabilizer H' of the radial half-line Δ^+ through the first vector e_1 of the canonical basis in $\mathbf{R}^n$. Moreover, H is a codimension one subgroup of H'.

By Lie [23] (see also [33, Part II, Chap. 6, Theorem 2.1]), the connected codimension one closed subgroups of H' are given by homomorphisms ψ from H' to $\mathbf{R}$, $\mathbf{Aff}$, or (some cover of) $PSL(2,\mathbf{R})$, where

$$\mathbf{Aff} = \left\{ \begin{pmatrix} a & b \\ 0 & 1/a \end{pmatrix} \quad : \quad a > 0 \right\}$$

is the group of affine transformations of the line. More precisely, H is (the component of the identity of) the inverse image by ψ of a codimension one subgroup, which is trivial in the case of $\mathbf{R}$, the subgroup of homotheties $(b = 0)$ in the case of $\mathbf{Aff}$ and the upper triangular subgroup in the case of $PSL(2,\mathbf{R})$. It is easy to see that there are no non-trivial homomorphisms of H' to $\mathbf{Aff}$. There are no non-trivial homomorphisms of H' to (any cover of) $PSL(2,\mathbf{R})$, except in the case $n = 3$. In this special case $n = 3$, one finds that H is the restricted upper-triangular group

$$U = \left\{ \begin{pmatrix} a & b & c \\ 0 & d & e \\ 0 & 0 & f \end{pmatrix} \quad : \quad a > 0 \right\},$$

which gives the compact flag manifold $SL(3,\mathbf{R})/U \cong S^3$. Finally, up to a multiplicative constant, there is a unique homomorphism from H' to $\mathbf{R}$:

$$\psi \colon (A_{ij}) \in H' \mapsto \ln A_{11} \in \mathbf{R}.$$

Note that here $H = \ker \psi$ is precisely the stabilizer $\mathrm{Stab}_{SL(n,\mathbf{R})}(e_1)$ of e_1 so that here $SL(n,\mathbf{R})/H$ is the homogeneous space $\mathbf{R}^n \backslash \{0\}$.

Thus we have dealt with the case where H is connected. Suppose that H is not connected, and let H_0 be its connected component of the identity. Now H_0 is a normal subgroup of H, and from above, by conjugation we may take H_0 to be either the group $\mathrm{Stab}_{SL(n,\mathbf{R})}(e_1)$, or the group U. If $H_0 = \mathrm{Stab}_{SL(n,\mathbf{R})}(e_1)$, notice that the normalizer of H_0 is the stabilizer H' of the radial half-line Δ^+. It follows that H/H_0 is a discrete subgroup of $\mathbf{R}$. If H/H_0 is finite, then $H/H_0 = \pm 1$ and so the quotient space is $\mathbf{R}^n \backslash \{0\}/\{\pm \mathrm{Id}\}$. If H/H_0 is infinite, then it is either infinite cyclic, or infinite cyclic cross $\mathbf{Z}/2\mathbf{Z}$, and in either case the quotient space is compact. If $H_0 = U$, the normalizer of H_0 is the full group $\overline{U}$ of upper-triangular matrices: there are 3 possibilities here, but in each case we get a compact quotient space.

This completes the proof of the theorem. $\square$

We now describe a useful method of extending an action of a subgroup to an action of the larger group. This method is very general and variations of it appear in various branches of mathematics: "induced module" in representation theory, "suspension" in dynamical systems, etc. In particular, it was used in an essential way in Schneider's classification of analytic $SL(2, \mathbf{R})$-actions on surfaces [37]. Suppose that H is a closed subgroup of a Lie group G and suppose that H acts continuously on a topological space F. So H acts diagonally on $G \times F$, where $g \in H \subset G$ acts on the first factor by right translation by g^{-1}. Let $E = (G \times F)/H$ denote the quotient space. So E fibres over the space G/H of left cosets of H, with fibre F. Now notice that G acts on $G \times F$ by left translation on the first factor, and this defines an action of G on E.

DEFINITION 3.6. The action of G on E just described is called the *suspension of the action of H on F.*

Notice that for such an action, there is a H-invariant subspace F' in E, which is H-equivariantly homeomorphic to F, and which has the property that $\mathrm{Stab}_H(x) = \mathrm{Stab}_G(x)$, for all $x \in F'$. Indeed, one can take $F' = \pi^{-1}(H)$, where $\pi\colon E \to G/H$ is the natural fibration. Given $f \in F$ and $g \in G$, let $[g,f]$ denote the image in E of (g,f) under the quotient map $G \times F \to E$. Then $\pi[g,f] = gH$, and $F' = \{[1,f] : f \in F\}\big(SL(n, \mathbf{R})\big)$.

Conversely, one has:

LEMMA 3.7. *Let H be a closed subgroup of a Lie group G. Suppose that G acts continuously on a topological space M and that there is a G-equivariant fibration $p\colon M \to G/H$. Then the G-action on M is conjugate to the suspension of the action of H on the fibre $F = p^{-1}(H)$. More precisely, if $E = (G \times F)/H$, then there is a G-equivariant homeomorphism from M to E which projects to the identity map on G/H.*

Proof. We define a function $\psi\colon M \to E$ as follows: for each $x \in M$ we set

$$\psi(x) = [g, g^{-1}(x)],$$

where $p(x) = gH$. Note that this makes sense since $g^{-1}(x) \in F$ and the definition of $\psi(x)$ doesn't depend upon the choice of g. By construction, ψ is G-equivariant and projects to the identity map on G/H. Finally, it is easy to see that ψ is a homeomorphism. $\square$

By Remark 2.2, $SO(n)$-actions of class C^0 on $(\mathbf{R}^m, 0)$ are not always linearizable. Despite this, we have the following result, which was proved for the cases $n \leq 3$ in [30, Chapter VI.6.5] and was conjectured therein for all n.

PROPOSITION 3.8. *Every faithful C^0-action of $SO(n)$ on $(\mathbf{R}^n, 0)$ is globally conjugate to the canonical linear action.*

Proof. By the proof of Theorem 3.5(a), the orbits of the $SO(n)$-action have dimension $\geq n-1$. In fact, there cannot be any $SO(n)$-orbit of dimension n, since otherwise it would be all of $\mathbf{R}^n\backslash\{0\}$, which is impossible, by the compactness of $SO(n)$. By the proof of Theorem 3.5(b), the only $SO(n)$-orbits of dimension $n-1$ are S^{n-1} and $\mathbf{R}P^{n-1}$, and the actions on them are conjugate to the canonical projective ones. In fact, for $n \geq 3$ there can be no orbit homeomorphic to $\mathbf{R}P^{n-1}$, because $\mathbf{R}P^{n-1}$ does not embed in $\mathbf{R}^n$ [6, Theorem 10.12]. So each orbit of $SO(n)$ is a $(n-1)$-dimensional sphere or a fixed point. It then follows from [30, ibid.] that 0 is the unique fixed point and there is a continuous ray γ beginning at 0 which meets each $SO(n)$-orbit exactly once.

First consider the $n=2$ case. Note that the $SO(2)$-action on $\mathbf{R}^2\backslash\{0\}$ is free. Indeed, let $g \in SO(2)$ and suppose that $x \in \mathbf{R}^2\backslash\{0\}$ belongs to the fixed point set $\mathrm{Fix}(g)$ of the action of g on $\mathbf{R}^2$. Then $\mathrm{Fix}(g)$ contains 0 as well as the entire orbit of x by $SO(2)$. By Eilenberg's theorem [9], since g is orientation preserving, the action of g on $\mathbf{R}^2$ is topologically conjugate to a rotation. So, as g has more than one fixed point, we must have $\mathrm{Fix}(g) = \mathbf{R}^2$. Hence, as the $SO(2)$-action on $\mathbf{R}^2$ is faithful by hypothesis, we have $g = \mathrm{Id}$, as claimed. Now define the map $\phi\colon \mathbf{R}^2 \to \mathbf{R}^2$ by setting

$$\phi\big(h\gamma(t)\big) = h\,.\begin{pmatrix} t \\ 0 \end{pmatrix},$$

for all $t \in [0,\infty)$, $h \in SO(2)$, where h acts on the left via the given $SO(2)$-action, and on the right by matrix multiplication. By construction, ϕ conjugates the given $SO(2)$-action to the canonical linear action.

Now suppose $n > 2$. Let $\{e_1, \dots, e_n\}$ denote the canonical basis of $\mathbf{R}^n$. Then, as in the proof in [30, ibid.], one may choose the ray γ to be comprised of fixed points of the restricted $SO(n-1)$-action, where here $SO(n-1)$ is the subgroup of $SO(n)$ which fixes the first basis vector e_1. So for each $x \in \mathbf{R}^n$, there is a unique number $t \in [0,\infty)$ and an element $g \in SO(n)$ such that $x = g\big(\gamma(t)\big)$. Moreover, for $x \in \mathbf{R}^n\backslash\{0\}$, the element g is unique modulo $SO(n-1)$. Consider the fibration

$$p\colon x \in \mathbf{R}^n\backslash\{0\} \mapsto g \in SO(n)/SO(n-1) \cong S^{n-1}.$$

Clearly p is $SO(n)$-equivariant. Notice that $p^{-1}\big(SO(n-1)\big) = \gamma\backslash\{0\} \cong \mathbf{R}$ and the $SO(n-1)$-action on this set is trivial. So, by Lemma 3.7, the action of $SO(n)$ on $\mathbf{R}^n\backslash\{0\}$ is conjugate to the action induced by the trivial action of $SO(n-1)$ on $\mathbf{R}$. That is, it is conjugate to the canonical action of $SO(n)$ on $\mathbf{R}^n\backslash\{0\}$. It remains to put back the origin. This can obviously be done equivariantly: one merely needs to verify that it can be done continuously. However, by averaging the flat metric on $\mathbf{R}^n$ by the original action of $SO(n)$, one may assume that the action is distance preserving. Thus, as t tends to 0, the $SO(n)$-orbits through $\gamma(t)$ converge uniformly to 0. So the continuity of the conjugation is clear. $\square$

We will also need the following:

LEMMA 3.9. *Let $n \geq 3$ and suppose that one has a C^0-action of $SL(n, \mathbf{R})$ on $(\mathbf{R}^n, 0)$ such that the restricted action of $SO(n)$ is the canonical linear action. Then locally the $SL(n, \mathbf{R})$-action preserves the radial lines.*

Proof. The key point is that two points of $\mathbf{R}^n$ lie in the same radial line if and only if they have the same stabilizer under the $SO(n)$-action. Let $x, y \in \mathbf{R}^n$ lie in the same radial line and let $g \in SL(n, \mathbf{R})$. So $\mathrm{Stab}_{SO(n)}(x) = \mathrm{Stab}_{SO(n)}(y)$ and we want to show that

$$\mathrm{Stab}_{SO(n)}\big(g(x)\big) = \mathrm{Stab}_{SO(n)}\big(g(y)\big).$$

Since the restricted action of $SO(n)$ is the canonical linear action, each orbit of $SL(n, \mathbf{R})$ in $\mathbf{R}^n\backslash\{0\}$ is either a round sphere centred at 0 or a spherical shell centred at 0. Suppose that our $SL(n, \mathbf{R})$-action on $\mathbf{R}^n$ has two spherical orbits, S_1 and S_2 say. By Theorem 3.5(b), the $SL(n, \mathbf{R})$-action on each sphere is the projective one. So there is an equivariant homeomorphism $\psi \colon S_1 \to S_2$. If $x \in S_1$ and $y = \psi(x) \in S_2$, we have $g(y) = \psi\big(g(x)\big)$ and as it is equivariant, ψ respects the stabilizers of the $SO(n)$-action. So $\mathrm{Stab}_{SO(n)}\big(g(y)\big) = \mathrm{Stab}_{SO(n)}\big(g(x)\big)$, as required (and ψ is just $\pm$ the radial projection of S_1 onto S_2).

By continuity, it remains to consider the case where x and y lie in the same open orbit of $SL(n, \mathbf{R})$; that is, suppose $y = h(x)$ for some $h \in SL(n, \mathbf{R})$. For all $f \in SL(n, \mathbf{R})$, one has $\mathrm{Stab}_{SO(n)}(x) = \mathrm{Stab}_{SO(n)}\big(f(x)\big)$ if and only if $f \in \mathrm{Norm}_{SL(n,\mathbf{R})}\big(\mathrm{Stab}_{SO(n)}(x)\big)$. So $h \in \mathrm{Norm}_{SL(n,\mathbf{R})}\big(\mathrm{Stab}_{SO(n)}(x)\big)$ and we need to show that $ghg^{-1} \in \mathrm{Norm}_{SL(n,\mathbf{R})}\big(\mathrm{Stab}_{SO(n)}\big(g(x)\big)\big)$. But if G is any group acting on a space X and H is a subgroup of G, then

$$\begin{aligned} g\big(\mathrm{Norm}_G\big(\mathrm{Stab}_H(x)\big)\big)g^{-1} &= \mathrm{Norm}_G\big(g\big(\mathrm{Stab}_H(x)g^{-1}\big)\big) \\ &= \mathrm{Norm}_G\big(\mathrm{Stab}_H\big(g(x)\big),\end{aligned}$$

for all $x \in X$ and $g \in G$, as we require.