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For $l \in L, v \in \mathbf{Q}^d$ we have $vF(ln^{-1})^{tr} = va^{-1}nFl^{tr}$ and hence $(L')^\# = L^\#an^{-1}$.

Since $L' \in \pi(L)$ one has $L' = L^\# \cap a^{-1}L$. Using this one obtains

$$Lan^{-2} = L'an^{-1} = L^\#an^{-1} \cap Ln^{-1} = (L')^\# \cap L' = L,$$

since $(L')^\#/L$ is the orthogonal complement of L'/L in $L^\#/L$ with respect to the induced quadratic form with values in \mathbf{Q}/\mathbf{Z} . So $a^{-1}n^2 \in G$.

Finally we check that $n \in N$. Let $g \in G$, then $n^{-1}gn$ is in $G = \text{Aut}(F)$ since $Ln^{-1}gn = L'gn = L'n = L$ and

$$n^{-1}gnFn^{tr}g^{tr}n^{-tr} = n^{-1}agFg^{tr}n^{-tr} = F. \quad \square$$

4. OBTAINING ELEMENTS OF N

Now we give examples as to how one may construct elements n of the normaliser N . To obtain similarities we are interested in $n \in N$ of determinant $\pm p^{d/2}$ for some (squarefree) natural number p such that $p^{-1}n^2 \in G$. The first method is an application of the normaliser principle to the situation (iii) described in Section 2:

PROPOSITION 4. *Let $U \trianglelefteq G$ be a normal subgroup of G and assume that the commuting algebra $K := C_{M_d(\mathbf{Q})}(U)$ is isomorphic to a number field. If $c \in K$ satisfies $c^2 = p \in \mathbf{Q}^*I_d$, then c lies in N .*

Proof. Since G normalises U , it acts by conjugation (and hence as Galois automorphisms) on the abelian number field K . Now let $c \in K$, with $c^2 =: p \in \mathbf{Q}^*I_d$ and $g \in G$. Then g stabilises the subfield $\mathbf{Q}[c]$ and hence $g^{-1}cg = \pm c$, which is equivalent to $c^{-1}gc = \pm g \in G$. Therefore $c \in N$, since we assumed that $-I_d \in G$. \square

The following construction described in [PIN 95] Proposition (II.4) also allows us to find elements of N .

For $i = 1, 2$ let $G_i \leq GL_{d_i}(\mathbf{Q})$ be finite rational irreducible matrix groups with commuting algebras $A_i \subseteq M_{d_i}(\mathbf{Q})$. Also let Q be a maximal common subalgebra of dimension z of A_1 and A_2 . Let $d := \frac{d_1d_2}{z}$ and view the G_i as subgroups of $G_1 \otimes_Q G_2 \leq GL_d(\mathbf{Q})$. If there exist elements $a_i \in N_{GL_d(\mathbf{Q})}(G_i)$

centralising G_j and a_j ($1 \leq i \neq j \leq 2$) and a squarefree natural number $p \neq 0$ such that $p^{-1}a_i^2 \in G_i$, the group

$$G := \langle G_1 \otimes_{\mathbf{Q}} G_2, p^{-1}a_1a_2 \rangle,$$

generated by the elements of $G_1 \otimes_{\mathbf{Q}} G_2$ and $p^{-1}a_1a_2$, is a finite subgroup of $GL_d(\mathbf{Q})$ containing $G_1 \otimes_{\mathbf{Q}} G_2$ as a subgroup of index 2.

For $d \leq 31$ and $p > 1$ we only need the case where a_2 is an element of the enveloping algebra of G_2 . Then G is denoted by $G_1 \otimes_{\mathbf{Q}}^{2(p)} G_2$ (or $G_1 \boxtimes_{\mathbf{Q}}^{2(p)} G_2$) according to whether a_1 is (or is not) a rational linear combination of elements of G_1 .

Using this notation one immediately has the following proposition.

PROPOSITION 5. *For $i = 1, 2$ the matrix a_i is an element of determinant $\pm p^{d/2}$ in the normaliser N of G .*

A common feature of the situations in Propositions 4 and 5 is that we extend the natural representation of G to a projective representation which is realisable as a linear representation over a quadratic extension of \mathbf{Q} .

PROPOSITION 6. *Let $G \trianglelefteq E$ be a supergroup containing G of index 2. Assume that $C_{M_d(\mathbf{Q})}(G) \cong \mathbf{Q}$ and that the natural character of G extends to E with character field $\mathbf{Q}[\sqrt{p}]$, where $p \in \mathbf{Z}$ is not a square. Then there exists $n \in N$ of determinant $\pm p^{d/2}$ with $p^{-1}n^2 \in G$.*

Proof. By Clifford theory one may extend the natural representation Δ of G to a representation $\delta_1 \otimes \delta_2 : E \rightarrow (\mathbf{Q}[\sqrt{p}] \otimes M_d(\mathbf{Q}))^*$, where δ_1 and δ_2 are projective representations $\delta_1(G) = \{1\}$ and $(\delta_2)|_G = \Delta$. Let $e \in E \setminus G$. Then

$$(\delta_1(e) \otimes \delta_2(e))^2 = \delta_1(e)^2 \otimes \delta_2(e)^2 = 1 \otimes \Delta(e^2),$$

since $e^2 \in G$. Therefore $\delta_1(e)^2 \in \mathbf{Q}$. Replacing $\delta_1(e)$ by a suitable rational multiple (and multiplying $\delta_2(e)$ by the inverse) one may assume that $\delta_1(e)^2 = p^{-1}$. Then $n := \delta_2(e)$ is an element of the normaliser N with the desired properties. \square

	$\text{Aut}(L)$	$\det(L)$	$\min(L)$	$ L_{\min} $	lattice sparse
4	$F_4 \tilde{\otimes} F_4 = [2_+^{1+8} \cdot O_8^+(2)]_{16}$	2^8	4	4320	+
6	$[(SL_2(9) \otimes_{\infty,3}^{2(3)} SL_2(9)) \cdot 2]_{16}$	3^8	4	720	+
9	$[(Sp_4(3) \circ C_3) \boxtimes_{\sqrt{-3}}^2 SL_2(3)]_{16}$	$2^8 \cdot 3^8$	6	960	+
14	$[2 \cdot \text{Alt}_{10}]_{16}$	5^8	6	2400	+
16	$[SL_2(5) \otimes_{\infty,2}^{2(2)} 2^{1+4'} \cdot \text{Alt}_5]_{16}$	$2^8 \cdot 5^8$	8	1200	+
19	$[SL_2(5) \otimes_{\infty,3}^{2(3)} SL_2(9)]_{16}$	$3^8 \cdot 5^8$	10	1440	+
21	$[SL_2(5) \otimes_{\infty,3}^{2(3)} (SL_2(3) \square^2 C_3)]_{16}$	$2^8 \cdot 3^8 \cdot 5^8$	12	480	+
25	$[2 \cdot \text{Alt}_7 \otimes_{\sqrt{-7}}^{2(3)} \tilde{S}_3]_{16}$	$3^8 \cdot 7^8$	12	1680	+
26	$[SL_2(7) \otimes_{\sqrt{-7}}^{2(3)} \tilde{S}_3]_{16}$	$3^8 \cdot 7^8$	10	336	$p \neq 2$
3	$[2 \cdot \text{Co}_1]_{24}$	1^{24}	4	196560	+
6	$[6 \cdot U_4(3) \cdot 2 \boxtimes_{\sqrt{-3}}^2 SL_2(3)]_{24}$	2^{12}	4	3024	$p \neq 3$
16	$[6 \cdot L_3(4) \cdot 2 \otimes^{2(2)} D_8]_{24}$	$2^{12} \cdot 3^{12}$	8	3024 + 7560	+
17	$[(SL_2(3) \circ C_4) \cdot 2 \boxtimes_{\sqrt{-1}}^{2(3)} U_3(3)]_{24}$	$2^{12} \cdot 3^{12}$	8	4536 + 6048	+
18	A_{24}	1^{24}	2	600	$p \neq 5$
22	$[2 \cdot J_2 \square^2 SL_2(5)]_{24}$	5^{12}	8	37800	+
35	$[L_2(7) \otimes^{2(2)} F_4]_{24}$	7^{12}	8	1008 + 3024	$p \neq 2$
40	$[SL_2(13) \square^{2(2)} SL_2(3)]_{24}$	13^{12}	12	$2 \cdot 2184 + 8736$	$p \neq 2$
42	$[6 \cdot \text{Alt}_7 : 2]_{24}$	2^{12}	4	3024	+
43	$[3 \cdot M_{10} \otimes_{\sqrt{-3}}^{2(2)} SL_2(3)]_{24}$	$2^{12} \cdot 5^{12}$	8	1080	$p \neq 3$
44	$[\text{Alt}_5 \otimes_{\sqrt{5}}^2 (C_3 \otimes^{2(2)} D_8)]_{24}$	$2^{12} \cdot 3^{12} \cdot 5^{12}$	16	$360 + 2 \cdot 720$	$p \neq 2$
45	$[3 \cdot M_{10} \otimes^{2(2)} D_8]_{24}$	$2^{12} \cdot 3^{12} \cdot 5^{12}$	16	$1080 + 1080$	+
64	$[SL_2(11) \otimes_{\sqrt{-11}}^{2(2)} SL_2(3)]_{24}$	$2^{12} \cdot 11^{12}$	12	1320	$p \neq 2$