

Zeitschrift:	L'Enseignement Mathématique
Herausgeber:	Commission Internationale de l'Enseignement Mathématique
Band:	43 (1997)
Heft:	1-2: L'ENSEIGNEMENT MATHÉMATIQUE
 Artikel:	THE NORMALISER ACTION AND STRONGLY MODULAR LATTICES
Autor:	Nebe, Gabriele
Kapitel:	3. Similarities Normalise
DOI:	https://doi.org/10.5169/seals-63272

Nutzungsbedingungen

Die ETH-Bibliothek ist die Anbieterin der digitalisierten Zeitschriften auf E-Periodica. Sie besitzt keine Urheberrechte an den Zeitschriften und ist nicht verantwortlich für deren Inhalte. Die Rechte liegen in der Regel bei den Herausgebern beziehungsweise den externen Rechteinhabern. Das Veröffentlichen von Bildern in Print- und Online-Publikationen sowie auf Social Media-Kanälen oder Webseiten ist nur mit vorheriger Genehmigung der Rechteinhaber erlaubt. [Mehr erfahren](#)

Conditions d'utilisation

L'ETH Library est le fournisseur des revues numérisées. Elle ne détient aucun droit d'auteur sur les revues et n'est pas responsable de leur contenu. En règle générale, les droits sont détenus par les éditeurs ou les détenteurs de droits externes. La reproduction d'images dans des publications imprimées ou en ligne ainsi que sur des canaux de médias sociaux ou des sites web n'est autorisée qu'avec l'accord préalable des détenteurs des droits. [En savoir plus](#)

Terms of use

The ETH Library is the provider of the digitised journals. It does not own any copyrights to the journals and is not responsible for their content. The rights usually lie with the publishers or the external rights holders. Publishing images in print and online publications, as well as on social media channels or websites, is only permitted with the prior consent of the rights holders. [Find out more](#)

Download PDF: 19.08.2025

ETH-Bibliothek Zürich, E-Periodica, <https://www.e-periodica.ch>

- (ii) $S = \{F \in M_d(\mathbf{Q}) \mid F = F^{tr}, F \text{ positive definite}\}$, the set of positive definite symmetric matrices, where x^{tr} denotes the transposed matrix of $x \in M_d(\mathbf{Q})$ and the action of H on S is $S \times H \rightarrow S$, $(F, h) \mapsto hFh^{tr}$. Then the set of G -fixed points is

$$\mathcal{F}_{>0}(G) := \{F \in S \mid gFg^{tr} = F \text{ for all } g \in G\}.$$

Note that $(\mathbf{R}_{>0})\mathcal{F}_{>0}(G)$ is the set of G -invariant Euclidean scalar products on \mathbf{R}^d . G is called *uniform*, if there is essentially one G -invariant Euclidean structure on \mathbf{R}^d , that is if $\mathcal{F}_{>0}(G) = \{aF \mid 0 < a \in \mathbf{Q}\}$ for some $F \in M_d(\mathbf{Q})$.

- (iii) $S = M_d(\mathbf{Q})$, and the action of H is conjugation: $S \times H \rightarrow S$, $(c, h) \mapsto h^{-1}ch$. Then the set of G -fixed points is the *commuting algebra* of G

$$C_{M_d(\mathbf{Q})}(G) := \{c \in M_d(\mathbf{Q}) \mid cg = gc \text{ for all } g \in G\}.$$

The following two remarks follow immediately from the normaliser principle.

REMARK 1. Assume that G is uniform and let $F \in \mathcal{F}_{>0}(G)$. Then for each $n \in N$, the matrix nFn^{tr} is also G -invariant and therefore $nFn^{tr} = (\det(n))^{2/d}F$. Hence n induces a similarity of F .

REMARK 2. For $n \in N$ and $L \in \mathcal{Z}(G)$, the lattice $Ln \in \mathcal{Z}(G)$ is also G -invariant.

3. SIMILARITIES NORMALISE

In this section we show that if G is the automorphism group of a (strongly modular) lattice L then the similarities between L and $L' \in \pi(L)$ are elements of N .

PROPOSITION 3. Let $G = \text{Aut}(F) \leq GL_d(\mathbf{Z})$ be the full automorphism group of a lattice L . Assume that L is an integral lattice. Let $L' \in \pi(L)$ and $n \in GL_d(\mathbf{Q})$ which induces a similarity from L' to L , i.e. $L'n = L$ and $nFn^{tr} = aF$, ($a \in \mathbf{N}$). Then $a^{-1}n^2 \in G$ and $n \in N$.

Proof. The matrix $a^{-1}n^2$ is clearly orthogonal with respect to F . Therefore to prove that $a^{-1}n^2 \in G$ we only have to show that $La^{-1}n^2 = L$. Now $L' = Ln^{-1}$, hence its dual lattice is

$$(L')^\# = \{v \in \mathbf{Q}^d \mid vF(ln^{-1})^{tr} \in \mathbf{Z} \text{ for all } l \in L\}.$$

For $l \in L, v \in \mathbf{Q}^d$ we have $vF(ln^{-1})^{tr} = va^{-1}nFl^{tr}$ and hence $(L')^\# = L^\#an^{-1}$.

Since $L' \in \pi(L)$ one has $L' = L^\# \cap a^{-1}L$. Using this one obtains

$$Lan^{-2} = L'an^{-1} = L^\#an^{-1} \cap Ln^{-1} = (L')^\# \cap L' = L,$$

since $(L')^\#/L$ is the orthogonal complement of L'/L in $L^\#/L$ with respect to the induced quadratic form with values in \mathbf{Q}/\mathbf{Z} . So $a^{-1}n^2 \in G$.

Finally we check that $n \in N$. Let $g \in G$, then $n^{-1}gn$ is in $G = \text{Aut}(F)$ since $Ln^{-1}gn = L'gn = L'n = L$ and

$$n^{-1}gnFn^{tr}g^{tr}n^{-tr} = n^{-1}agFg^{tr}n^{-tr} = F.$$

□

4. OBTAINING ELEMENTS OF N

Now we give examples as to how one may construct elements n of the normaliser N . To obtain similarities we are interested in $n \in N$ of determinant $\pm p^{d/2}$ for some (squarefree) natural number p such that $p^{-1}n^2 \in G$. The first method is an application of the normaliser principle to the situation (iii) described in Section 2:

PROPOSITION 4. *Let $U \trianglelefteq G$ be a normal subgroup of G and assume that the commuting algebra $K := C_{M_d(\mathbf{Q})}(U)$ is isomorphic to a number field. If $c \in K$ satisfies $c^2 = p \in \mathbf{Q}^*I_d$, then c lies in N .*

Proof. Since G normalises U , it acts by conjugation (and hence as Galois automorphisms) on the abelian number field K . Now let $c \in K$, with $c^2 =: p \in \mathbf{Q}^*I_d$ and $g \in G$. Then g stabilises the subfield $\mathbf{Q}[c]$ and hence $g^{-1}cg = \pm c$, which is equivalent to $c^{-1}gc = \pm g \in G$. Therefore $c \in N$, since we assumed that $-I_d \in G$. □

The following construction described in [PIN 95] Proposition (II.4) also allows us to find elements of N .

For $i = 1, 2$ let $G_i \leq GL_{d_i}(\mathbf{Q})$ be finite rational irreducible matrix groups with commuting algebras $A_i \subseteq M_{d_i}(\mathbf{Q})$. Also let Q be a maximal common subalgebra of dimension z of A_1 and A_2 . Let $d := \frac{d_1d_2}{z}$ and view the G_i as subgroups of $\mathop{\otimes}_Q G_1 \otimes G_2 \leq GL_d(\mathbf{Q})$. If there exist elements $a_i \in N_{GL_d(\mathbf{Q})}(G_i)$