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of G in $GL_d(\mathbf{Q})$ to test strong modularity of L . In the next section we derive some methods for explicitly constructing elements of $N_{GL_d(\mathbf{Q})}(G)$.

Every finite subgroup of $GL_d(\mathbf{Q})$ is a subgroup of the automorphism group of an integral lattice. In particular the maximal finite subgroups of $GL_d(\mathbf{Q})$ are automorphism groups of distinguished lattices. A subgroup of $GL_d(\mathbf{Q})$ is called rational irreducible if it does not preserve a proper subspace $\neq \{0\}$ of \mathbf{Q}^d . The rational irreducible maximal finite, abbreviated to *r.i.m.f.*, subgroups of $GL_d(\mathbf{Q})$ are classified for $d < 32$ (cf. [PIN 95], [NeP 95], [Neb 95], [Neb 96], [Neb 96a]). Their invariant lattices provide many examples of strongly modular lattices. The following theorem is proved by applying the methods derived in Section 4.

THEOREM. *In dimension $d < 32$, all even lattices $L \subseteq \mathbf{R}^d$ that are preserved by a r.i.m.f. group and satisfy $L^\# / L \cong (\mathbf{Z} / l\mathbf{Z})^{d/2}$ for some $l \in \mathbf{N}$ are strongly modular, except for the lattices of the r.i.m.f. group $[\pm \text{Alt}_6 . 2^2]_{16}$ in $GL_{16}(\mathbf{Q})$ (cf. [NeP 95]).*

2. PRELIMINARIES AND NOTATION

The main strategy in this paper is the application of the following *normaliser principle*.

Let G be a group acting on a set S , H a subgroup of the group of transformations of S . Then the normaliser of G in H acts on the set of G -orbits.

In our situation $G = \text{Aut}(L)$ is the automorphism group of an integral lattice L in the Euclidean space $\mathbf{R}L \cong \mathbf{R}^d$. By writing the action of G on $\mathbf{R}L$ with respect to a \mathbf{Z} -basis (b_1, \dots, b_d) of L , G becomes a finite subgroup of $GL_d(\mathbf{Z})$. Then $G = \text{Aut}(F) = \{g \in GL_d(\mathbf{Z}) \mid gFg^{tr} = F\}$ where F is the Gram matrix $F = ((b_i, b_j))_{i,j=1}^d$ of L .

For the rest of this article let $H = GL_d(\mathbf{Q})$, $G \leq H$, be a finite subgroup of H , and let $N := N_H(G)$ be its normaliser. We also assume that G contains the negative unit matrix, $-I_d \in G$.

We apply the normaliser principle to the following three situations.

- (i) $S = \{L \subseteq \mathbf{Q}^d \mid L = \sum_{i=1}^d \mathbf{Z}b_i \text{ for a basis } (b_1, \dots, b_d) \text{ of } \mathbf{Q}^d\}$, the set of \mathbf{Z} -lattices of rank d in \mathbf{Q}^d , and the action of H on S is right multiplication: $S \times H \rightarrow S$, $(L, h) \mapsto Lh := \{lh \mid l \in L\}$. Then the set of G -fixed points is

$$\mathcal{Z}(G) := \{L \in S \mid Lg = L \text{ for all } g \in G\},$$

the set of G -invariant lattices.

- (ii) $S = \{F \in M_d(\mathbf{Q}) \mid F = F^{tr}, F \text{ positive definite}\}$, the set of positive definite symmetric matrices, where x^{tr} denotes the transposed matrix of $x \in M_d(\mathbf{Q})$ and the action of H on S is $S \times H \rightarrow S$, $(F, h) \mapsto hFh^{tr}$. Then the set of G -fixed points is

$$\mathcal{F}_{>0}(G) := \{F \in S \mid gFg^{tr} = F \text{ for all } g \in G\}.$$

Note that $(\mathbf{R}_{>0})\mathcal{F}_{>0}(G)$ is the set of G -invariant Euclidean scalar products on \mathbf{R}^d . G is called *uniform*, if there is essentially one G -invariant Euclidean structure on \mathbf{R}^d , that is if $\mathcal{F}_{>0}(G) = \{aF \mid 0 < a \in \mathbf{Q}\}$ for some $F \in M_d(\mathbf{Q})$.

- (iii) $S = M_d(\mathbf{Q})$, and the action of H is conjugation: $S \times H \rightarrow S$, $(c, h) \mapsto h^{-1}ch$. Then the set of G -fixed points is the *commuting algebra* of G

$$C_{M_d(\mathbf{Q})}(G) := \{c \in M_d(\mathbf{Q}) \mid cg = gc \text{ for all } g \in G\}.$$

The following two remarks follow immediately from the normaliser principle.

REMARK 1. Assume that G is uniform and let $F \in \mathcal{F}_{>0}(G)$. Then for each $n \in N$, the matrix nFn^{tr} is also G -invariant and therefore $nFn^{tr} = (\det(n))^{2/d}F$. Hence n induces a similarity of F .

REMARK 2. For $n \in N$ and $L \in \mathcal{Z}(G)$, the lattice $Ln \in \mathcal{Z}(G)$ is also G -invariant.

3. SIMILARITIES NORMALISE

In this section we show that if G is the automorphism group of a (strongly modular) lattice L then the similarities between L and $L' \in \pi(L)$ are elements of N .

PROPOSITION 3. Let $G = \text{Aut}(F) \leq GL_d(\mathbf{Z})$ be the full automorphism group of a lattice L . Assume that L is an integral lattice. Let $L' \in \pi(L)$ and $n \in GL_d(\mathbf{Q})$ which induces a similarity from L' to L , i.e. $L'n = L$ and $nFn^{tr} = aF$, ($a \in \mathbf{N}$). Then $a^{-1}n^2 \in G$ and $n \in N$.

Proof. The matrix $a^{-1}n^2$ is clearly orthogonal with respect to F . Therefore to prove that $a^{-1}n^2 \in G$ we only have to show that $La^{-1}n^2 = L$. Now $L' = Ln^{-1}$, hence its dual lattice is

$$(L')^\# = \{v \in \mathbf{Q}^d \mid vF(ln^{-1})^{tr} \in \mathbf{Z} \text{ for all } l \in L\}.$$