7. Arithmetic intersection theory

We recall here the generalization of Arakelov theory to higher dimensions due to Gillet and Soulé. Our main references are [GS1], [GS2] and the exposition in [SABK]. For A an abelian group, $A_{\mathbf{Q}}$ denotes $A\otimes_{\mathbf{Z}}\mathbf{Q}$. Let X be an *arithmetic scheme over* $\mathbf{Z}$, by which we mean a regular scheme, projective and flat over $\operatorname{Spec}\mathbf{Z}$. For $p\geqslant 0$, let $X^{(p)}$ be the set of integral subschemes of X of codimension p and $Z^p(X)$ be the group of codimension p cycles on X. The p-th Chow group of X: $CH^p(X):=Z^p(X)/R^p(X)$, where $R^p(X)$ is the subgroup of $Z^p(X)$ generated by the cycles $\operatorname{div}f$, $f\in k(x)^*$, $x\in X^{(p-1)}$. Let $CH(X)=\bigoplus_p CH^p(X)$. If X is smooth over $\operatorname{Spec}\mathbf{Z}$, then the methods of [F] can be used to give $CH(X)$ the structure of a commutative ring. In general one has a product structure on $CH(X)_{\mathbf{Q}}$ after tensoring with $\mathbf{Q}$.

Let $D^{p,p}\big(X(\mathbf{C})\big)$ denote the space of complex currents of type (p,p) on $X(\mathbf{C})$, and $F_\infty : X(\mathbf{C})\to X(\mathbf{C})$ the involution induced by complex conjugation. Let $D^{p,p}(X_{\mathbf{R}})$ (resp. $A^{p,p}(X_{\mathbf{R}})$) be the subspace of $D^{p,p}\big(X(\mathbf{C})\big)$ (resp. $A^{p,p}(X(\mathbf{C}))$) generated by real currents (resp. forms) T such that $F_\infty^*T=(-1)^pT$; denote by $\widetilde{D}^{p,p}(X_{\mathbf{R}})$ and $\widetilde{A}^{p,p}(X_{\mathbf{R}})$ the respective images in $\widetilde{D}^{p,p}\big(X(\mathbf{C})\big)$ and $\widetilde{A}^{p,p}\big(X(\mathbf{C})\big)$.

An *arithmetic cycle* on X of codimension p is a pair (Z,g_Z) in the group $Z^p(X)\bigoplus\widetilde{D}^{p-1,p-1}(X_{\mathbf{R}})$, where g_Z is a *Green current* for $Z(\mathbf{C})$, i.e. a current such that $dd^cg_Z+\delta_{Z(\mathbf{C})}$ is represented by a smooth form. The group of arithmetic cycles is denoted by $\widehat{Z}^p(X)$. If $x\in X^{(p-1)}$ and $f\in k(x)^*$, we let $\widehat{\operatorname{div}}f$ denote the arithmetic cycle $(\operatorname{div}f,[-\log|f_{\mathbf{C}}|^2\cdot\delta_{x(\mathbf{C})}])$.

The *p-th arithmetic Chow group of X*: $\widehat{CH}^p(X):=\widehat{Z}^p(X)/\widehat{R}^p(X)$, where $\widehat{R}^p(X)$ is the subgroup of $\widehat{Z}^p(X)$ generated by the cycles $\widehat{\operatorname{div}}f$, $f\in k(x)^*$, $x\in X^{(p-1)}$. Let $\widehat{CH}(X)=\bigoplus_p\widehat{CH}^p(X)$.

We have the following canonical morphisms of abelian groups:

$$\zeta:\widehat{CH}^p(X)\longrightarrow CH^p(X),\qquad \big[(Z,g_Z)\big]\longmapsto[Z],$$
$$\omega:\widehat{CH}^p(X)\longrightarrow \operatorname{Ker}d\cap\operatorname{Ker}d^c\big(\subset A^{p,p}(X_{\mathbf{R}})\big),\qquad \big[(Z,g_Z)\big]\longmapsto dd^cg_Z+\delta_{Z(\mathbf{C})},$$
$$a:\widetilde{A}^{p-1,p-1}(X_{\mathbf{R}})\longrightarrow\widehat{CH}^p(X),\qquad \eta\longmapsto\big[(0,\eta)\big].$$

One can define a pairing $\widehat{CH}^p(X)\otimes\widehat{CH}^q(X)\to\widehat{CH}^{p+q}(X)_{\mathbf{Q}}$ which turns $\widehat{CH}(X)_{\mathbf{Q}}$ into a commutative graded unitary $\mathbf{Q}$-algebra. The maps ζ, ω are $\mathbf{Q}$-algebra homomorphisms. If X is smooth over $\mathbf{Z}$ one does not have to tensor with $\mathbf{Q}$. The definition of this pairing is difficult; the construction uses the *star product* of Green currents, which in turn relies upon Hironaka's resolution of singularities to get to the case of divisors. The functor $\widehat{CH}^p(X)$ is contravariant in X, and covariant for proper maps which are smooth on the generic fiber.

Choose a Kähler form ω_0 on $X(\mathbf{C})$ such that $F_\infty^*\omega_0 = -\omega_0$ (this is equivalent to requiring that the corresponding Kähler metric is invariant under F_∞). It is natural to utilize the theory of harmonic forms on X in the study of Green currents on $X(\mathbf{C})$. Following [GS1], we call the pair $\overline{X} = (X, \omega_0)$ an *Arakelov variety*. By the Hodge decomposition theorem, we have $A^{p,p}(X_{\mathbf{R}}) = \mathcal{H}^{p,p}(X_{\mathbf{R}}) \oplus \operatorname{Im} d \oplus \operatorname{Im} d^*$, where $\mathcal{H}^{p,p}(X_{\mathbf{R}}) = \operatorname{Ker}\Delta \subset A^{p,p}(X)$ denotes the space of harmonic (with respect to ω_0) (p,p) forms α on $X(\mathbf{C})$ such that $F_\infty^*\alpha = (-1)^p\alpha$. The subgroup $CH^p(\overline{X}) := \omega^{-1}\big(\mathcal{H}^{p,p}(X_{\mathbf{R}})\big)$ of $\widehat{CH}^p(X)$ is called the *p-th Arakelov Chow group of X*. Let $CH(\overline{X}) = \bigoplus_{p\geqslant 0} CH^p(\overline{X})$. $CH^p(\overline{X})$ is a direct summand of $\widehat{CH}^p(X)$, and there is an exact sequence

$$(11) \qquad CH^{p,p-1}(X) \xrightarrow{\ \rho\ } \mathcal{H}^{p-1,p-1}(X_{\mathbf{R}}) \xrightarrow{\ a\ } CH^p(\overline{X}) \xrightarrow{\ \zeta\ } CH^p(X) \longrightarrow 0\,.$$

In the above sequence the group $CH^{p,p-1}(X)$ is defined as the $E_2^{p,1-p}$ term of a certain spectral sequence used by Quillen to calculate the higher algebraic K-theory of X, and the map ρ coincides with the Beilinson regulator map (cf. [G] and [GS1], 3.5).

If $\mathcal{H}(X_{\mathbf{R}}) = \bigoplus_p \mathcal{H}^{p,p}(X_{\mathbf{R}})$ is a subring of $\bigoplus_p A^{p,p}(X_{\mathbf{R}})$, for example if $X(\mathbf{C})$ is a curve, an abelian variety or a hermitian symmetric space (e.g. a Grassmannian), then $CH(\overline{X})_{\mathbf{Q}}$ is a subring of $\widehat{CH}(X)_{\mathbf{Q}}$. This is not the case in general; for example it fails to be true for the complete flag varieties.

Arakelov [A] introduced the group $CH^1(\overline{X})$, where $\overline{X} = (X, g_0)$ is an arithmetic surface with the metric g_0 on the Riemann surface $X(\mathbf{C})$ given by $\frac{i}{2g}\sum \omega_j \wedge \overline{\omega}_j$. Here g is the genus of $X(\mathbf{C})$ and $\{\omega_j\}$ for $1 \leqslant j \leqslant g$ is an orthonormal basis of the space of holomorphic one forms on $X(\mathbf{C})$.

A *hermitian vector bundle* $\overline{E} = (E, h)$ on an arithmetic scheme X is an algebraic vector bundle E on X such that the induced holomorphic vector bundle $E(\mathbf{C})$ on $X(\mathbf{C})$ has a hermitian metric h, which is invariant under complex conjugation, i.e. $F_\infty^*(h) = h$.

To any hermitian vector bundle one can attach characteristic classes $\widehat{\phi}(\overline{E}) \in \widehat{CH}(X)_{\mathbf{Q}}$, for any $\phi \in I(n, \mathbf{Q})$, where $n = \operatorname{rk} E$. For example, we have *arithmetic Chern classes* $\widehat{c}_k(\overline{E}) \in \widehat{CH}^k(X)$. Some basic properties of these classes are:

(1) $\widehat{c}_0(\overline{E}) = 1$ and $\widehat{c}_p(\overline{E}) = 0$ for $k > \operatorname{rk} E$.

(2) The form $\omega\big(\widehat{c}_k(\overline{E})\big) = c_k(\overline{E}) \in A^{k,k}(X_{\mathbf{R}})$ is the k-th Chern form of the hermitian bundle $\overline{E(\mathbf{C})}$.

(3) $\zeta\big(\widehat{c}_k(\bar{E})\big) = c_k(E) \in CH^k(X)$.

(4) $f^*\widehat{c}_k(\bar{E}) = \widehat{c}_k(f^*\bar{E})$, for every morphism $f : X \to Y$ of regular schemes, projective and flat over $\mathbf{Z}$.

(5) If $\bar{L}$ is a hermitian line bundle, $\widehat{c}_1(\bar{L})$ is the class of $(\mathrm{div}(s), -\log\|s\|^2)$ for any rational section s of L.

Analogous properties are satisfied by $\widehat{\phi}$ for any $\phi \in I(n, \mathbf{Q})$ (see [GS2], Th. 4.1). The most relevant property of these characteristic classes is their behaviour in short exact sequences: if

$$\bar{\mathcal{E}}:\ 0 \to \bar{S} \to \bar{E} \to \bar{Q} \to 0$$

is such a sequence of hermitian vector bundles over X, then

$$\widehat{\phi}(\bar{S} \oplus \bar{Q}) - \widehat{\phi}(\bar{E}) = a\big(\widetilde{\phi}(\bar{\mathcal{E}})\big)\,. \tag{12}$$

Relation (12) is the main tool for calculating intersection products of classes in $\widehat{CH}(X)$ that come from characteristic classes of vector bundles. Combining it with the results of §4 and §5 gives immediate consequences for such intersections. For example, we have

COROLLARY 4. *Let $\bar{\mathcal{E}} : 0 \to \bar{S} \to \bar{E} \to \bar{Q} \to 0$ be a short exact sequence of hermitian vector bundles over an arithmetic scheme X. Assume that the metrics on $S(\mathbf{C})$, $Q(\mathbf{C})$ are induced from that on $E(\mathbf{C})$.*

(a) *If $\overline{E(\mathbf{C})}$ is flat, then*

(1) $\widehat{p_\lambda}(\bar{S} \oplus \bar{Q}) = \widehat{p_\lambda}(\bar{E})$, *if λ has length > 1, and*

(2) $\widehat{p_k}(\bar{S}) + \widehat{p_k}(\bar{Q}) - \widehat{p_k}(\bar{E}) = k\mathcal{H}_{k-1}a\big(p_{k-1}(\bar{Q})\big)$, $\forall k \geqslant 1$,

in the arithmetic Chow group $\widehat{CH}(X)_{\mathbf{Q}}$.

(b) *If $\bar{E} = \bar{L}^{\oplus n}$ for some hermitian line bundle $\bar{L}$ and $\omega = c_1\big(\overline{L(\mathbf{C})}\big)$, then*

$$\widehat{c}(\bar{S})\,\widehat{c}(\bar{Q}) - \widehat{c}(\bar{E}) = \sum_{i,j}(-1)^j \binom{n}{i} (\mathcal{H}_n - \mathcal{H}_{n-i} + \mathcal{H}_j)a\big(\omega^i p_j(\bar{Q})\big)\,,$$

in the arithmetic Chow group $\widehat{CH}(X)$.