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## 7. ARITHMETIC INTERSECTION THEORY

We recall here the generalization of Arakelov theory to higher dimensions due to Gillet and Soulé. Our main references are [GS1], [GS2] and the exposition in [SABK]. For  $A$  an abelian group,  $A_{\mathbf{Q}}$  denotes  $A \otimes_{\mathbf{Z}} \mathbf{Q}$ . Let  $X$  be an *arithmetic scheme over  $\mathbf{Z}$* , by which we mean a regular scheme, projective and flat over  $\text{Spec } \mathbf{Z}$ . For  $p \geq 0$ , let  $X^{(p)}$  be the set of integral subschemes of  $X$  of codimension  $p$  and  $Z^p(X)$  be the group of codimension  $p$  cycles on  $X$ . The  $p$ -th Chow group of  $X$ :  $CH^p(X) := Z^p(X)/R^p(X)$ , where  $R^p(X)$  is the subgroup of  $Z^p(X)$  generated by the cycles  $\text{div } f$ ,  $f \in k(x)^*$ ,  $x \in X^{(p-1)}$ . Let  $CH(X) = \bigoplus_p CH^p(X)$ . If  $X$  is smooth over  $\text{Spec } \mathbf{Z}$ , then the methods of [F] can be used to give  $CH(X)$  the structure of a commutative ring. In general one has a product structure on  $CH(X)_{\mathbf{Q}}$  after tensoring with  $\mathbf{Q}$ .

Let  $D^{p,p}(X(\mathbf{C}))$  denote the space of complex currents of type  $(p,p)$  on  $X(\mathbf{C})$ , and  $F_{\infty} : X(\mathbf{C}) \rightarrow X(\mathbf{C})$  the involution induced by complex conjugation. Let  $D^{p,p}(X_{\mathbf{R}})$  (resp.  $A^{p,p}(X_{\mathbf{R}})$ ) be the subspace of  $D^{p,p}(X(\mathbf{C}))$  (resp.  $A^{p,p}(X(\mathbf{C}))$ ) generated by real currents (resp. forms)  $T$  such that  $F_{\infty}^* T = (-1)^p T$ ; denote by  $\tilde{D}^{p,p}(X_{\mathbf{R}})$  and  $\tilde{A}^{p,p}(X_{\mathbf{R}})$  the respective images in  $\tilde{D}^{p,p}(X(\mathbf{C}))$  and  $\tilde{A}^{p,p}(X(\mathbf{C}))$ .

An *arithmetic cycle* on  $X$  of codimension  $p$  is a pair  $(Z, g_Z)$  in the group  $Z^p(X) \oplus \tilde{D}^{p-1,p-1}(X_{\mathbf{R}})$ , where  $g_Z$  is a *Green current* for  $Z(\mathbf{C})$ , i.e. a current such that  $dd^c g_Z + \delta_{Z(\mathbf{C})}$  is represented by a smooth form. The group of arithmetic cycles is denoted by  $\widehat{Z}^p(X)$ . If  $x \in X^{(p-1)}$  and  $f \in k(x)^*$ , we let  $\widehat{\text{div}} f$  denote the arithmetic cycle  $(\text{div } f, [-\log |f_{\mathbf{C}}|^2 \cdot \delta_{x(\mathbf{C})}])$ .

The  $p$ -th *arithmetic Chow group* of  $X$ :  $\widehat{CH}^p(X) := \widehat{Z}^p(X)/\widehat{R}^p(X)$ , where  $\widehat{R}^p(X)$  is the subgroup of  $\widehat{Z}^p(X)$  generated by the cycles  $\widehat{\text{div}} f$ ,  $f \in k(x)^*$ ,  $x \in X^{(p-1)}$ . Let  $\widehat{CH}(X) = \bigoplus_p \widehat{CH}^p(X)$ .

We have the following canonical morphisms of abelian groups:

$$\begin{aligned} \zeta : \widehat{CH}^p(X) &\longrightarrow CH^p(X), & [(Z, g_Z)] &\longmapsto [Z], \\ \omega : \widehat{CH}^p(X) &\longrightarrow \text{Ker } d \cap \text{Ker } d^c (\subset A^{p,p}(X_{\mathbf{R}})), & [(Z, g_Z)] &\longmapsto dd^c g_Z + \delta_{Z(\mathbf{C})}, \\ a : \tilde{A}^{p-1,p-1}(X_{\mathbf{R}}) &\longrightarrow \widehat{CH}^p(X), & \eta &\longmapsto [(0, \eta)]. \end{aligned}$$

One can define a pairing  $\widehat{CH}^p(X) \otimes \widehat{CH}^q(X) \rightarrow \widehat{CH}^{p+q}(X)_{\mathbf{Q}}$  which turns  $\widehat{CH}(X)_{\mathbf{Q}}$  into a commutative graded unitary  $\mathbf{Q}$ -algebra. The maps  $\zeta$ ,  $\omega$  are  $\mathbf{Q}$ -algebra homomorphisms. If  $X$  is smooth over  $\mathbf{Z}$  one does not have to tensor with  $\mathbf{Q}$ . The definition of this pairing is difficult; the construction uses the *star product* of Green currents, which in turn relies upon Hironaka's

resolution of singularities to get to the case of divisors. The functor  $\widehat{CH}^p(X)$  is contravariant in  $X$ , and covariant for proper maps which are smooth on the generic fiber.

Choose a Kähler form  $\omega_0$  on  $X(\mathbf{C})$  such that  $F_\infty^* \omega_0 = -\omega_0$  (this is equivalent to requiring that the corresponding Kähler metric is invariant under  $F_\infty$ ). It is natural to utilize the theory of harmonic forms on  $X$  in the study of Green currents on  $X(\mathbf{C})$ . Following [GS1], we call the pair  $\bar{X} = (X, \omega_0)$  an *Arakelov variety*. By the Hodge decomposition theorem, we have  $A^{p,p}(X_{\mathbf{R}}) = \mathcal{H}^{p,p}(X_{\mathbf{R}}) \oplus \text{Im } d \oplus \text{Im } d^*$ , where  $\mathcal{H}^{p,p}(X_{\mathbf{R}}) = \text{Ker } \Delta \subset A^{p,p}(X)$  denotes the space of harmonic (with respect to  $\omega_0$ )  $(p, p)$  forms  $\alpha$  on  $X(\mathbf{C})$  such that  $F_\infty^* \alpha = (-1)^p \alpha$ . The subgroup  $CH^p(\bar{X}) := \omega^{-1}(\mathcal{H}^{p,p}(X_{\mathbf{R}}))$  of  $\widehat{CH}^p(X)$  is called the  $p$ -th *Arakelov Chow group of  $X$* . Let  $CH(\bar{X}) = \bigoplus_{p \geq 0} CH^p(\bar{X})$ .  $CH^p(\bar{X})$  is a direct summand of  $\widehat{CH}^p(X)$ , and there is an exact sequence

$$(11) \quad CH^{p,p-1}(X) \xrightarrow{\rho} \mathcal{H}^{p-1,p-1}(X_{\mathbf{R}}) \xrightarrow{a} CH^p(\bar{X}) \xrightarrow{\zeta} CH^p(X) \longrightarrow 0.$$

In the above sequence the group  $CH^{p,p-1}(X)$  is defined as the  $E_2^{p,1-p}$  term of a certain spectral sequence used by Quillen to calculate the higher algebraic  $K$ -theory of  $X$ , and the map  $\rho$  coincides with the Beilinson regulator map (cf. [G] and [GS1], 3.5).

If  $\mathcal{H}(X_{\mathbf{R}}) = \bigoplus_p \mathcal{H}^{p,p}(X_{\mathbf{R}})$  is a subring of  $\bigoplus_p A^{p,p}(X_{\mathbf{R}})$ , for example if  $X(\mathbf{C})$  is a curve, an abelian variety or a hermitian symmetric space (e.g. a Grassmannian), then  $CH(\bar{X})_{\mathbf{Q}}$  is a subring of  $\widehat{CH}(X)_{\mathbf{Q}}$ . This is not the case in general; for example it fails to be true for the complete flag varieties.

Arakelov [A] introduced the group  $CH^1(\bar{X})$ , where  $\bar{X} = (X, g_0)$  is an arithmetic surface with the metric  $g_0$  on the Riemann surface  $X(\mathbf{C})$  given by  $\frac{i}{2g} \sum \omega_j \wedge \bar{\omega}_j$ . Here  $g$  is the genus of  $X(\mathbf{C})$  and  $\{\omega_j\}$  for  $1 \leq j \leq g$  is an orthonormal basis of the space of holomorphic one forms on  $X(\mathbf{C})$ .

A *hermitian vector bundle*  $\bar{E} = (E, h)$  on an arithmetic scheme  $X$  is an algebraic vector bundle  $E$  on  $X$  such that the induced holomorphic vector bundle  $E(\mathbf{C})$  on  $X(\mathbf{C})$  has a hermitian metric  $h$ , which is invariant under complex conjugation, i.e.  $F_\infty^*(h) = h$ .

To any hermitian vector bundle one can attach characteristic classes  $\widehat{\phi}(\bar{E}) \in \widehat{CH}(X)_{\mathbf{Q}}$ , for any  $\phi \in I(n, \mathbf{Q})$ , where  $n = \text{rk } E$ . For example, we have *arithmetic Chern classes*  $\widehat{c}_k(\bar{E}) \in \widehat{CH}^k(X)$ . Some basic properties of these classes are:

- (1)  $\widehat{c}_0(\bar{E}) = 1$  and  $\widehat{c}_p(\bar{E}) = 0$  for  $k > \text{rk } E$ .
- (2) The form  $\omega(\widehat{c}_k(\bar{E})) = c_k(\bar{E}) \in A^{k,k}(X_{\mathbf{R}})$  is the  $k$ -th Chern form of the hermitian bundle  $\overline{E(\mathbf{C})}$ .

$$(3) \quad \zeta(\widehat{c}_k(\bar{E})) = c_k(E) \in CH^k(X).$$

(4)  $f^*\widehat{c}_k(\bar{E}) = \widehat{c}_k(f^*\bar{E})$ , for every morphism  $f : X \rightarrow Y$  of regular schemes, projective and flat over  $\mathbf{Z}$ .

(5) If  $\bar{L}$  is a hermitian line bundle,  $\widehat{c}_1(\bar{L})$  is the class of  $(\text{div}(s), -\log \|s\|^2)$  for any rational section  $s$  of  $L$ .

Analogous properties are satisfied by  $\widehat{\phi}$  for any  $\phi \in I(n, \mathbf{Q})$  (see [GS2], Th. 4.1). The most relevant property of these characteristic classes is their behaviour in short exact sequences: if

$$\bar{\mathcal{E}} : 0 \rightarrow \bar{S} \rightarrow \bar{E} \rightarrow \bar{Q} \rightarrow 0$$

is such a sequence of hermitian vector bundles over  $X$ , then

$$(12) \quad \widehat{\phi}(\bar{S} \oplus \bar{Q}) - \widehat{\phi}(\bar{E}) = a(\widetilde{\phi}(\bar{\mathcal{E}})).$$

Relation (12) is the main tool for calculating intersection products of classes in  $\widehat{CH}(X)$  that come from characteristic classes of vector bundles. Combining it with the results of §4 and §5 gives immediate consequences for such intersections. For example, we have

**COROLLARY 4.** *Let  $\bar{\mathcal{E}} : 0 \rightarrow \bar{S} \rightarrow \bar{E} \rightarrow \bar{Q} \rightarrow 0$  be a short exact sequence of hermitian vector bundles over an arithmetic scheme  $X$ . Assume that the metrics on  $S(\mathbf{C})$ ,  $Q(\mathbf{C})$  are induced from that on  $E(\mathbf{C})$ .*

(a) *If  $\overline{E(\mathbf{C})}$  is flat, then*

- (1)  $\widehat{p}_\lambda(\bar{S} \oplus \bar{Q}) = \widehat{p}_\lambda(\bar{E})$ , if  $\lambda$  has length  $> 1$ , and
- (2)  $\widehat{p}_k(\bar{S}) + \widehat{p}_k(\bar{Q}) - \widehat{p}_k(\bar{E}) = k\mathcal{H}_{k-1}a(p_{k-1}(\bar{Q}))$ ,  $\forall k \geq 1$ ,

*in the arithmetic Chow group  $\widehat{CH}(X)_{\mathbf{Q}}$ .*

(b) *If  $\bar{E} = \bar{L}^{\oplus n}$  for some hermitian line bundle  $\bar{L}$  and  $\omega = c_1(\overline{L(\mathbf{C})})$ , then*

$$\widehat{c}(\bar{S})\widehat{c}(\bar{Q}) - \widehat{c}(\bar{E}) = \sum_{i,j} (-1)^j \binom{n}{i} (\mathcal{H}_n - \mathcal{H}_{n-i} + \mathcal{H}_j) a(\omega^i p_j(\bar{Q})),$$

*in the arithmetic Chow group  $\widehat{CH}(X)$ .*