

Zeitschrift: L'Enseignement Mathématique
Herausgeber: Commission Internationale de l'Enseignement Mathématique
Band: 43 (1997)
Heft: 1-2: L'ENSEIGNEMENT MATHÉMATIQUE

Artikel: BOTT-CHERN FORMS AND ARITHMETIC INTERSECTIONS
Autor: Tamvakis, Harry
Kapitel: 5. $S \rightarrow E \rightarrow Q$
 $0 \rightarrow \text{FLAT}$
DOI: <https://doi.org/10.5169/seals-63270>

Nutzungsbedingungen

Die ETH-Bibliothek ist die Anbieterin der digitalisierten Zeitschriften auf E-Periodica. Sie besitzt keine Urheberrechte an den Zeitschriften und ist nicht verantwortlich für deren Inhalte. Die Rechte liegen in der Regel bei den Herausgebern beziehungsweise den externen Rechteinhabern. Das Veröffentlichen von Bildern in Print- und Online-Publikationen sowie auf Social Media-Kanälen oder Webseiten ist nur mit vorheriger Genehmigung der Rechteinhaber erlaubt. [Mehr erfahren](#)

Conditions d'utilisation

L'ETH Library est le fournisseur des revues numérisées. Elle ne détient aucun droit d'auteur sur les revues et n'est pas responsable de leur contenu. En règle générale, les droits sont détenus par les éditeurs ou les détenteurs de droits externes. La reproduction d'images dans des publications imprimées ou en ligne ainsi que sur des canaux de médias sociaux ou des sites web n'est autorisée qu'avec l'accord préalable des détenteurs des droits. [En savoir plus](#)

Terms of use

The ETH Library is the provider of the digitised journals. It does not own any copyrights to the journals and is not responsible for their content. The rights usually lie with the publishers or the external rights holders. Publishing images in print and online publications, as well as on social media channels or websites, is only permitted with the prior consent of the rights holders. [Find out more](#)

Download PDF: 27.04.2026

ETH-Bibliothek Zürich, E-Periodica, <https://www.e-periodica.ch>

$$5. \quad 0 \rightarrow \bar{S} \rightarrow \bar{E} \rightarrow \bar{Q} \rightarrow 0 \text{ WITH } \bar{E} \text{ FLAT}$$

Throughout this section we will assume that the hermitian vector bundle \bar{E} is *flat*, i.e. that $K_E = 0$. As before, the metrics h_S and h_Q will be induced from the metric on E . Define the *harmonic numbers* $\mathcal{H}_k = \sum_{i=1}^k \frac{1}{i}$, $\mathcal{H}_0 = 0$.

Let λ be a partition of k (we denote this by $\lambda \vdash k$). Recall that the polynomials $\{p_\lambda : \lambda \vdash k\}$ form a \mathbf{Q} -basis for the vector space of symmetric homogeneous polynomials in x_1, \dots, x_n of degree k . The following result computes the Bott-Chern form corresponding to any such invariant polynomial:

THEOREM 3. *The Bott-Chern class $\widetilde{p}_\lambda(\bar{\mathcal{E}})$ in $\widetilde{A}(X)$ is the class of*

- (i) $k\mathcal{H}_{k-1}p_{k-1}(\bar{Q})$, if $\lambda = k = (k, 0, 0, \dots, 0)$
- (ii) 0, otherwise.

Proof. Let us first compute $\widetilde{p}_k(\bar{\mathcal{E}})$ for $p_k(A) = \text{Tr}(A^k)$. Since $K_E = 0$, the deformed matrix $K(u) = (1-u)K_{S \oplus Q}$, where $K_{S \oplus Q} = \begin{pmatrix} K_S & 0 \\ 0 & K_Q \end{pmatrix}$. Since

$$\int_0^1 \frac{(1-u)^{k-1} - 1}{u} du = - \int_0^1 \frac{t^{k-1} - 1}{t-1} dt = -\mathcal{H}_{k-1},$$

we obtain

$$\widetilde{p}_k(\bar{\mathcal{E}}) = -\mathcal{H}_{k-1}p'_k(K_{S \oplus Q}; J_r) = -k\mathcal{H}_{k-1} \text{Tr}(K_S^{k-1}) = -k\mathcal{H}_{k-1}p_{k-1}(\bar{S}).$$

Now since $p_k(\bar{S} \oplus \bar{Q}) - p_k(\bar{E})$ is exact, $p_k(\bar{E}) = 0$ and $p_k(\bar{S} \oplus \bar{Q}) = p_k(\bar{S}) + p_k(\bar{Q})$, we conclude that $p_k(\bar{S}) = -p_k(\bar{Q})$ in $\widetilde{A}(X)$, for each $k \geq 1$. This proves (i).

Let $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_m)$ be a partition ($m \geq 2$). Proposition 1 implies that

$$\widetilde{p}_\lambda(\bar{\mathcal{E}}) = \widetilde{p}_{\lambda_1}(\bar{\mathcal{E}})p_{\lambda_2} \cdots p_{\lambda_m}(\bar{S} \oplus \bar{Q}).$$

But $\widetilde{p}_{\lambda_1}(\bar{\mathcal{E}})$ is a closed form (by (i)), and $p_{\lambda_2} \cdots p_{\lambda_m}(\bar{S} \oplus \bar{Q})$ is an exact form. Thus $\widetilde{p}_\lambda(\bar{\mathcal{E}})$ is exact, and so vanishes in $\widetilde{A}(X)$. \square

It follows from Theorem 3 that for any $\phi \in I(n)$, the Bott-Chern form $\widetilde{\phi}(\bar{\mathcal{E}})$ is a linear combination of homogeneous components of the Chern character form $ch(\bar{Q})$. In [Ma], Theorem 3.4.1 we find the calculation

$$(4) \quad \tilde{c}_k(\bar{\mathcal{E}}) = \mathcal{H}_{k-1} \sum_{i=0}^{k-1} i c_i(\bar{\mathcal{S}}) c_{k-1-i}(\bar{\mathcal{Q}})$$

for the Chern forms \tilde{c}_k . Our result gives the following

$$\text{PROPOSITION 3.} \quad \tilde{c}_k(\bar{\mathcal{E}}) = (-1)^{k-1} \mathcal{H}_{k-1} p_{k-1}(\bar{\mathcal{Q}}).$$

Proof. By Newton's identity (2) we have

$$(5) \quad \tilde{p}_k - \widetilde{c_1 p_{k-1}} + \widetilde{c_2 p_{k-2}} - \cdots + (-1)^k k \tilde{c}_k = 0.$$

Reasoning as in Theorem 3, we see that if ϕ and ψ are two homogeneous invariant polynomials of positive degree, then $\widetilde{\phi\psi}(\bar{\mathcal{E}}) = 0$ in $\tilde{A}(X)$. Thus (5) gives $\tilde{c}_k(\bar{\mathcal{E}}) = \frac{(-1)^{k-1}}{k} \tilde{p}_k(\bar{\mathcal{E}}) = (-1)^{k-1} \mathcal{H}_{k-1} p_{k-1}(\bar{\mathcal{Q}})$. \square

REMARK. The result of Proposition 3 agrees with (4), i.e. $(-1)^k p_k(\bar{\mathcal{Q}}) = \sum_{i=0}^k i c_i(\bar{\mathcal{S}}) c_{k-i}(\bar{\mathcal{Q}})$ in $\tilde{A}(X)$. To see this, let $h(t) = \sum c_i(\bar{\mathcal{S}}) t^i$, $g(t) = \sum c_j(\bar{\mathcal{Q}}) t^j$, and $f(t) = \sum i c_i(\bar{\mathcal{S}}) t^i$. Then $h(t)g(t) = 1$ in $\tilde{A}(X)[t]$, and $f(t) = th'(t)$. Choose formal variables $\{x_\alpha\}_{1 \leq \alpha \leq r}$ and set $c_i(\bar{\mathcal{S}}) = e_i(x_1, \dots, x_r)$, so that $h(t) = \prod_{\alpha} (1 + x_\alpha t)$. Then $f(t) = \sum_{\alpha} t x_\alpha \prod_{\beta \neq \alpha} (1 + x_\beta t)$. Thus

$$\begin{aligned} f(t)g(t) &= \frac{f(t)}{h(t)} = \sum_{\alpha} \frac{x_\alpha t}{1 + x_\alpha t} \\ &= r - \sum_{\alpha} \frac{1}{1 + x_\alpha t} = r - \sum_{\alpha, i} (-1)^i x_\alpha^i t^i = r - \sum_i (-1)^i p_i(\bar{\mathcal{S}}) t^i \\ &= r + \sum_i (-1)^i p_i(\bar{\mathcal{Q}}) t^i. \end{aligned}$$

Comparing coefficients of t^k on both sides gives the result.

We can use Theorem 3 to calculate $\widetilde{\phi}(\bar{\mathcal{E}})$ for $\phi \in I(n)_k$: it is enough to find the coefficient of the power sum p_k when ϕ is expressed as a linear combination of $\{p_\lambda\}_{\lambda \vdash k}$ in $\Lambda(n, \mathbf{Q})$. For example, we have

COROLLARY 2. $\widetilde{ch}(\bar{\mathcal{E}}) = \sum_k \mathcal{H}_k ch_k(\bar{\mathcal{Q}})$, where ch_k denotes the k -th homogeneous component of the Chern character form.

COROLLARY 3. Let λ be a partition of k and s_λ the corresponding Schur polynomial in $\Lambda(n, \mathbf{Q})$. Then $\widetilde{s}_\lambda(\overline{\mathcal{E}}) = 0$ unless λ is a hook $\lambda^i = (i, 1, 1, \dots, 1)$, in which case $\widetilde{s}_{\lambda^i}(\overline{\mathcal{E}}) = (-1)^{k-i} \mathcal{H}_{k-1} p_{k-1}(\overline{\mathcal{Q}})$.

Proof. The proof is based on the Frobenius formula

$$s_\lambda = \frac{1}{k!} \sum_{\sigma \in \mathcal{S}_k} \chi_\lambda(\sigma) p(\sigma)$$

where (σ) denotes the partition of k determined by the cycle structure of σ (cf. [M], §I.7). By the above remark, $\widetilde{s}_\lambda(\overline{\mathcal{E}}) = \chi_\lambda((12\dots k)) \mathcal{H}_{k-1} p_{k-1}(\overline{\mathcal{Q}})$. Using the combinatorial rule for computing χ_λ found in [M], p. 117, Example 5, we obtain

$$\chi_\lambda((12\dots k)) = \begin{cases} (-1)^{k-i}, & \text{if } \lambda = \lambda^i \text{ is a hook} \\ 0, & \text{otherwise.} \end{cases} \quad \square$$

The most natural instance of a sequence $\overline{\mathcal{E}}$ with \overline{E} flat is the classifying sequence over the Grassmannian $G(r, n)$. As we shall see in §8, the calculation of Bott-Chern forms for this sequence leads to a presentation of the Arakelov Chow ring of the arithmetic Grassmannian over $\text{Spec } \mathbf{Z}$.

6. CALCULATIONS WHEN \overline{E} IS PROJECTIVELY FLAT

We will now generalize the results of the last section to the case where E is *projectively flat*, i.e. the curvature matrix K_E of \overline{E} is a multiple of the identity matrix: $K_E = \omega Id_n$. This is true if $E = \overline{L}^{\oplus n}$ for some hermitian line bundle \overline{L} , with $\omega = c_1(\overline{L})$ the first Chern form of \overline{L} .

The Bott-Chern forms (for the induced metrics) are always closed in this case as well, and will be expressed in terms of characteristic classes of the bundles involved. However this seems to be the most general case where this phenomenon occurs.

The key observation is that for projectively flat bundles, the curvature matrix $K_E = \omega Id_n$ in *any* local trivialization. Thus we have

$$K(u) = \left(\begin{array}{c|c} (1-u)K_S + u\omega Id_r & 0 \\ \hline 0 & (1-u)K_Q + u\omega Id_s \end{array} \right)$$

where $s = n - r$ denotes the rank of Q . Now Theorem 2 gives