

# 5. $\mathbb{0} \rightarrow \bar{S} \rightarrow \bar{E} \rightarrow \bar{Q} \rightarrow \mathbb{0}$ WITH $\bar{E}$ FLAT

Objektyp: **Chapter**

Zeitschrift: **L'Enseignement Mathématique**

Band (Jahr): **43 (1997)**

Heft 1-2: **L'ENSEIGNEMENT MATHÉMATIQUE**

PDF erstellt am: **26.09.2024**

## **Nutzungsbedingungen**

Die ETH-Bibliothek ist Anbieterin der digitalisierten Zeitschriften. Sie besitzt keine Urheberrechte an den Inhalten der Zeitschriften. Die Rechte liegen in der Regel bei den Herausgebern.

Die auf der Plattform e-periodica veröffentlichten Dokumente stehen für nicht-kommerzielle Zwecke in Lehre und Forschung sowie für die private Nutzung frei zur Verfügung. Einzelne Dateien oder Ausdrucke aus diesem Angebot können zusammen mit diesen Nutzungsbedingungen und den korrekten Herkunftsbezeichnungen weitergegeben werden.

Das Veröffentlichen von Bildern in Print- und Online-Publikationen ist nur mit vorheriger Genehmigung der Rechteinhaber erlaubt. Die systematische Speicherung von Teilen des elektronischen Angebots auf anderen Servern bedarf ebenfalls des schriftlichen Einverständnisses der Rechteinhaber.

## **Haftungsausschluss**

Alle Angaben erfolgen ohne Gewähr für Vollständigkeit oder Richtigkeit. Es wird keine Haftung übernommen für Schäden durch die Verwendung von Informationen aus diesem Online-Angebot oder durch das Fehlen von Informationen. Dies gilt auch für Inhalte Dritter, die über dieses Angebot zugänglich sind.

$$5. \quad 0 \rightarrow \bar{S} \rightarrow \bar{E} \rightarrow \bar{Q} \rightarrow 0 \text{ WITH } \bar{E} \text{ FLAT}$$

Throughout this section we will assume that the hermitian vector bundle  $\bar{E}$  is *flat*, i.e. that  $K_E = 0$ . As before, the metrics  $h_S$  and  $h_Q$  will be induced from the metric on  $E$ . Define the *harmonic numbers*  $\mathcal{H}_k = \sum_{i=1}^k \frac{1}{i}$ ,  $\mathcal{H}_0 = 0$ .

Let  $\lambda$  be a partition of  $k$  (we denote this by  $\lambda \vdash k$ ). Recall that the polynomials  $\{p_\lambda : \lambda \vdash k\}$  form a  $\mathbf{Q}$ -basis for the vector space of symmetric homogeneous polynomials in  $x_1, \dots, x_n$  of degree  $k$ . The following result computes the Bott-Chern form corresponding to any such invariant polynomial:

**THEOREM 3.** *The Bott-Chern class  $\widetilde{p}_\lambda(\bar{\mathcal{E}})$  in  $\widetilde{A}(X)$  is the class of*

- (i)  $k\mathcal{H}_{k-1}p_{k-1}(\bar{Q})$ , if  $\lambda = k = (k, 0, 0, \dots, 0)$
- (ii) 0, otherwise.

*Proof.* Let us first compute  $\widetilde{p}_k(\bar{\mathcal{E}})$  for  $p_k(A) = \text{Tr}(A^k)$ . Since  $K_E = 0$ , the deformed matrix  $K(u) = (1-u)K_{S \oplus Q}$ , where  $K_{S \oplus Q} = \begin{pmatrix} K_S & 0 \\ 0 & K_Q \end{pmatrix}$ . Since

$$\int_0^1 \frac{(1-u)^{k-1} - 1}{u} du = - \int_0^1 \frac{t^{k-1} - 1}{t-1} dt = -\mathcal{H}_{k-1},$$

we obtain

$$\widetilde{p}_k(\bar{\mathcal{E}}) = -\mathcal{H}_{k-1}p'_k(K_{S \oplus Q}; J_r) = -k\mathcal{H}_{k-1} \text{Tr}(K_S^{k-1}) = -k\mathcal{H}_{k-1}p_{k-1}(\bar{S}).$$

Now since  $p_k(\bar{S} \oplus \bar{Q}) - p_k(\bar{E})$  is exact,  $p_k(\bar{E}) = 0$  and  $p_k(\bar{S} \oplus \bar{Q}) = p_k(\bar{S}) + p_k(\bar{Q})$ , we conclude that  $p_k(\bar{S}) = -p_k(\bar{Q})$  in  $\widetilde{A}(X)$ , for each  $k \geq 1$ . This proves (i).

Let  $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_m)$  be a partition ( $m \geq 2$ ). Proposition 1 implies that

$$\widetilde{p}_\lambda(\bar{\mathcal{E}}) = \widetilde{p}_{\lambda_1}(\bar{\mathcal{E}})p_{\lambda_2} \cdots p_{\lambda_m}(\bar{S} \oplus \bar{Q}).$$

But  $\widetilde{p}_{\lambda_1}(\bar{\mathcal{E}})$  is a closed form (by (i)), and  $p_{\lambda_2} \cdots p_{\lambda_m}(\bar{S} \oplus \bar{Q})$  is an exact form. Thus  $\widetilde{p}_\lambda(\bar{\mathcal{E}})$  is exact, and so vanishes in  $\widetilde{A}(X)$ .  $\square$

It follows from Theorem 3 that for any  $\phi \in I(n)$ , the Bott-Chern form  $\widetilde{\phi}(\bar{\mathcal{E}})$  is a linear combination of homogeneous components of the Chern character form  $ch(\bar{Q})$ . In [Ma], Theorem 3.4.1 we find the calculation

$$(4) \quad \tilde{c}_k(\bar{\mathcal{E}}) = \mathcal{H}_{k-1} \sum_{i=0}^{k-1} i c_i(\bar{\mathcal{S}}) c_{k-1-i}(\bar{\mathcal{Q}})$$

for the Chern forms  $\tilde{c}_k$ . Our result gives the following

$$\text{PROPOSITION 3.} \quad \tilde{c}_k(\bar{\mathcal{E}}) = (-1)^{k-1} \mathcal{H}_{k-1} p_{k-1}(\bar{\mathcal{Q}}).$$

*Proof.* By Newton's identity (2) we have

$$(5) \quad \tilde{p}_k - \widetilde{c_1 p_{k-1}} + \widetilde{c_2 p_{k-2}} - \cdots + (-1)^k k \tilde{c}_k = 0.$$

Reasoning as in Theorem 3, we see that if  $\phi$  and  $\psi$  are two homogeneous invariant polynomials of positive degree, then  $\widetilde{\phi\psi}(\bar{\mathcal{E}}) = 0$  in  $\tilde{A}(X)$ . Thus (5) gives  $\tilde{c}_k(\bar{\mathcal{E}}) = \frac{(-1)^{k-1}}{k} \tilde{p}_k(\bar{\mathcal{E}}) = (-1)^{k-1} \mathcal{H}_{k-1} p_{k-1}(\bar{\mathcal{Q}})$ .  $\square$

REMARK. The result of Proposition 3 agrees with (4), i.e.  $(-1)^k p_k(\bar{\mathcal{Q}}) = \sum_{i=0}^k i c_i(\bar{\mathcal{S}}) c_{k-i}(\bar{\mathcal{Q}})$  in  $\tilde{A}(X)$ . To see this, let  $h(t) = \sum c_i(\bar{\mathcal{S}}) t^i$ ,  $g(t) = \sum c_j(\bar{\mathcal{Q}}) t^j$ , and  $f(t) = \sum i c_i(\bar{\mathcal{S}}) t^i$ . Then  $h(t)g(t) = 1$  in  $\tilde{A}(X)[t]$ , and  $f(t) = th'(t)$ . Choose formal variables  $\{x_\alpha\}_{1 \leq \alpha \leq r}$  and set  $c_i(\bar{\mathcal{S}}) = e_i(x_1, \dots, x_r)$ , so that  $h(t) = \prod_{\alpha} (1 + x_\alpha t)$ . Then  $f(t) = \sum_{\alpha} t x_\alpha \prod_{\beta \neq \alpha} (1 + x_\beta t)$ . Thus

$$\begin{aligned} f(t)g(t) &= \frac{f(t)}{h(t)} = \sum_{\alpha} \frac{x_\alpha t}{1 + x_\alpha t} \\ &= r - \sum_{\alpha} \frac{1}{1 + x_\alpha t} = r - \sum_{\alpha, i} (-1)^i x_\alpha^i t^i = r - \sum_i (-1)^i p_i(\bar{\mathcal{S}}) t^i \\ &= r + \sum_i (-1)^i p_i(\bar{\mathcal{Q}}) t^i. \end{aligned}$$

Comparing coefficients of  $t^k$  on both sides gives the result.

We can use Theorem 3 to calculate  $\widetilde{\phi}(\bar{\mathcal{E}})$  for  $\phi \in I(n)_k$ : it is enough to find the coefficient of the power sum  $p_k$  when  $\phi$  is expressed as a linear combination of  $\{p_\lambda\}_{\lambda \vdash k}$  in  $\Lambda(n, \mathbf{Q})$ . For example, we have

COROLLARY 2.  $\widetilde{ch}(\bar{\mathcal{E}}) = \sum_k \mathcal{H}_k ch_k(\bar{\mathcal{Q}})$ , where  $ch_k$  denotes the  $k$ -th homogeneous component of the Chern character form.

COROLLARY 3. Let  $\lambda$  be a partition of  $k$  and  $s_\lambda$  the corresponding Schur polynomial in  $\Lambda(n, \mathbf{Q})$ . Then  $\widetilde{s}_\lambda(\overline{\mathcal{E}}) = 0$  unless  $\lambda$  is a hook  $\lambda^i = (i, 1, 1, \dots, 1)$ , in which case  $\widetilde{s}_{\lambda^i}(\overline{\mathcal{E}}) = (-1)^{k-i} \mathcal{H}_{k-1} p_{k-1}(\overline{\mathcal{Q}})$ .

*Proof.* The proof is based on the Frobenius formula

$$s_\lambda = \frac{1}{k!} \sum_{\sigma \in \mathcal{S}_k} \chi_\lambda(\sigma) p(\sigma)$$

where  $(\sigma)$  denotes the partition of  $k$  determined by the cycle structure of  $\sigma$  (cf. [M], §I.7). By the above remark,  $\widetilde{s}_\lambda(\overline{\mathcal{E}}) = \chi_\lambda((12\dots k)) \mathcal{H}_{k-1} p_{k-1}(\overline{\mathcal{Q}})$ . Using the combinatorial rule for computing  $\chi_\lambda$  found in [M], p. 117, Example 5, we obtain

$$\chi_\lambda((12\dots k)) = \begin{cases} (-1)^{k-i}, & \text{if } \lambda = \lambda^i \text{ is a hook} \\ 0, & \text{otherwise.} \end{cases} \quad \square$$

The most natural instance of a sequence  $\overline{\mathcal{E}}$  with  $\overline{E}$  flat is the classifying sequence over the Grassmannian  $G(r, n)$ . As we shall see in §8, the calculation of Bott-Chern forms for this sequence leads to a presentation of the Arakelov Chow ring of the arithmetic Grassmannian over  $\text{Spec } \mathbf{Z}$ .

## 6. CALCULATIONS WHEN $\overline{E}$ IS PROJECTIVELY FLAT

We will now generalize the results of the last section to the case where  $E$  is *projectively flat*, i.e. the curvature matrix  $K_E$  of  $\overline{E}$  is a multiple of the identity matrix:  $K_E = \omega Id_n$ . This is true if  $E = \overline{L}^{\oplus n}$  for some hermitian line bundle  $\overline{L}$ , with  $\omega = c_1(\overline{L})$  the first Chern form of  $\overline{L}$ .

The Bott-Chern forms (for the induced metrics) are always closed in this case as well, and will be expressed in terms of characteristic classes of the bundles involved. However this seems to be the most general case where this phenomenon occurs.

The key observation is that for projectively flat bundles, the curvature matrix  $K_E = \omega Id_n$  in *any* local trivialization. Thus we have

$$K(u) = \left( \begin{array}{c|c} (1-u)K_S + u\omega Id_r & 0 \\ \hline 0 & (1-u)K_Q + u\omega Id_s \end{array} \right)$$

where  $s = n - r$  denotes the rank of  $Q$ . Now Theorem 2 gives