

**Zeitschrift:** L'Enseignement Mathématique  
**Herausgeber:** Commission Internationale de l'Enseignement Mathématique  
**Band:** 43 (1997)  
**Heft:** 1-2: L'ENSEIGNEMENT MATHÉMATIQUE

**Artikel:** BOTT-CHERN FORMS AND ARITHMETIC INTERSECTIONS  
**Autor:** Tamvakis, Harry  
**Kapitel:** 4. Calculating Bott-Chern Forms  
**DOI:** <https://doi.org/10.5169/seals-63270>

### **Nutzungsbedingungen**

Die ETH-Bibliothek ist die Anbieterin der digitalisierten Zeitschriften auf E-Periodica. Sie besitzt keine Urheberrechte an den Zeitschriften und ist nicht verantwortlich für deren Inhalte. Die Rechte liegen in der Regel bei den Herausgebern beziehungsweise den externen Rechteinhabern. Das Veröffentlichen von Bildern in Print- und Online-Publikationen sowie auf Social Media-Kanälen oder Webseiten ist nur mit vorheriger Genehmigung der Rechteinhaber erlaubt. [Mehr erfahren](#)

### **Conditions d'utilisation**

L'ETH Library est le fournisseur des revues numérisées. Elle ne détient aucun droit d'auteur sur les revues et n'est pas responsable de leur contenu. En règle générale, les droits sont détenus par les éditeurs ou les détenteurs de droits externes. La reproduction d'images dans des publications imprimées ou en ligne ainsi que sur des canaux de médias sociaux ou des sites web n'est autorisée qu'avec l'accord préalable des détenteurs des droits. [En savoir plus](#)

### **Terms of use**

The ETH Library is the provider of the digitised journals. It does not own any copyrights to the journals and is not responsible for their content. The rights usually lie with the publishers or the external rights holders. Publishing images in print and online publications, as well as on social media channels or websites, is only permitted with the prior consent of the rights holders. [Find out more](#)

**Download PDF:** 27.04.2026

**ETH-Bibliothek Zürich, E-Periodica, <https://www.e-periodica.ch>**

## 4. CALCULATING BOTT-CHERN FORMS

In this section we will consider an exact sequence

$$\bar{\mathcal{E}} : 0 \rightarrow \bar{S} \rightarrow \bar{E} \rightarrow \bar{Q} \rightarrow 0,$$

where the metrics on  $\bar{S}$  and  $\bar{Q}$  are induced from the metric on  $E$ . Let  $r, n$  be the ranks of the bundles  $S$  and  $E$ . Let  $\phi \in I(n)$  be homogeneous of degree  $k$ . We will formulate a theorem for calculating the Bott-Chern form  $\tilde{\phi}(\bar{\mathcal{E}})$ . This result follows from the work of Bott-Chern, Cowen, Bismut and Gillet-Soulé.

Let  $\phi'$  be defined as in §2. For any two matrices  $A, B \in M_n(\mathbf{C})$  set

$$\phi'(A; B) := \sum_{i=1}^k \phi'(A, A, \dots, A, B_{(i)}, A, \dots, A),$$

where the index  $i$  means that  $B$  is in the  $i$ -th position.

Choose a local orthonormal frame  $s = (s_1, s_2, \dots, s_n)$  of  $E$  such that the first  $r$  elements generate  $S$ , and let  $K(\bar{S}), K(\bar{E})$  and  $K(\bar{Q})$  be the curvature matrices of  $\bar{S}, \bar{Q}$  and  $\bar{E}$  with respect to  $s$ . Let  $K_S = \frac{i}{2\pi}K(\bar{S}), K_E = \frac{i}{2\pi}K(\bar{E})$  and  $K_Q = \frac{i}{2\pi}K(\bar{Q})$ . The matrix  $K_E$  has the form

$$K_E = \left( \begin{array}{c|c} K_{11} & K_{12} \\ \hline K_{21} & K_{22} \end{array} \right)$$

where  $K_{11}$  is an  $r \times r$  submatrix. Also consider the matrices

$$K_0 = \left( \begin{array}{c|c} K_S & 0 \\ \hline K_{21} & K_Q \end{array} \right) \quad \text{and} \quad J_r = \left( \begin{array}{c|c} Id_r & 0 \\ \hline 0 & 0 \end{array} \right).$$

Let  $u$  be a formal variable and  $K(u) := uK_E + (1 - u)K_0$ . Finally, let  $\phi^!(u) = \phi'(K(u); J_r)$ . We then have the following

**THEOREM 2.**

$$(3) \quad \tilde{\phi}(\bar{\mathcal{E}}) = \int_0^1 \frac{\phi^!(u) - \phi^!(0)}{u} du.$$

*Proof.* We prove that  $\tilde{\phi}(\bar{\mathcal{E}})$  as defined above satisfies axioms (i)-(iii) of Theorem 1. The main step is the first axiom; this was essentially done in [BC] §4, when  $\phi = c$  is the total Chern class. In the form (3) (again for the total Chern class), the equation was given by Cowen in [C1] and [C2], while

simplifying Bott and Chern's proof. We follow both sources in sketching a proof of this more general result.

Let  $h$  and  $h_Q$  denote the metrics on  $E$  and  $Q$  respectively. Define the orthogonal projections  $P_1 : \bar{E} \rightarrow \bar{S}$  and  $P_2 : \bar{E} \rightarrow \bar{Q}$  and put  $h_u(v, v') = uh(P_1v, P_1v') + h(P_2v, P_2v')$  for  $v, v' \in E_x$  and  $0 < u \leq 1$ . Then  $h_u$  is a hermitian norm,  $h_1 = h$  and  $h_u \rightarrow h_Q$  as  $u \rightarrow 0$ . Let  $K(E, h_u)$  be the curvature matrix of  $(E, h_u)$  relative to the holomorphic frame  $s$  defined above. Proposition 3.1 of [C2] proves that  $\frac{i}{2\pi}K(E, h_u) = K(u)$ . It follows from Proposition 3.28 of [BC] that for  $0 < t \leq 1$ ,

$$\phi(E, h_t) - \phi(E, h) = dd^c \int_t^1 \frac{\phi'(K(u); J_r)}{u} du.$$

If we could set  $t = 0$  we would be done; however, the integral will not be convergent in general. Note that  $K(u) = K_0 + uK_1$ , where  $K_1 \in A^{1,1}(X, \text{End}(E))$  is independent of  $u$ . Therefore it will suffice to show that  $\phi'(K_0; J_r)$  is a closed form, so that it can be deleted from the integral. For this we may assume that  $\phi = p_\lambda$  is a product of power sums,  $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_m)$  a partition. Then

$$\begin{aligned} p'_\lambda(K_0; J_r) &= \sum_{i=1}^m \text{Tr}(K_S)^{\lambda_i-1} \prod_{j \neq i} (\text{Tr}(K_S) + \text{Tr}(K_Q))^{\lambda_j} \\ &= \sum_{i=1}^m p_{\lambda_i-1}(\bar{S}) \prod_{j \neq i} p_{\lambda_j}(\bar{S} \oplus \bar{Q}) \end{aligned}$$

is certainly a closed form.

This proves axioms (i) and (iii); axiom (ii) is easily checked as well.  $\square$

REMARK. A similar deformation to the one in [C2] was used by Deligne in [D], 5.11 for a calculation involving the Chern character form. Special cases of Theorem 2 have been used in the literature before, see for example [GS2] Prop. 5.3, [GSZ] 2.2.3 and [Ma] Theorem 3.3.1.

We deduce some simple but useful calculations:

COROLLARY 1.

- (a)  $\tilde{c}_1^k(\bar{\mathcal{E}}) = 0$  for all  $k \geq 1$  and  $\tilde{c}_m(\bar{\mathcal{E}}) = 0$  for all  $m > \text{rk} E$ .  
 (b)  $\tilde{p}_2(\bar{\mathcal{E}}) = 2(\text{Tr} K_{11} - c_1(\bar{S}))$  and  $\tilde{c}_2(\bar{\mathcal{E}}) = c_1(\bar{S}) - \text{Tr} K_{11}$ .

*Proof.* (a)  $c_1^!(u)$  is independent of  $u$ ; hence  $\tilde{c}_1(\bar{\mathcal{E}}) = 0$ . The result for higher powers of  $c_1$  follows from Proposition 1. In addition,  $\tilde{c}_m(\bar{\mathcal{E}}) = 0$  for  $m > \text{rk} E$  is an immediate consequence of the definition.

(b) Using the bilinear form  $p'_2$  described previously, we find  $p_2^1(u) = 2(u \operatorname{Tr} K_{11} + (1-u)c_1(\bar{S}))$ , so

$$\begin{aligned} \tilde{p}_2(\bar{\mathcal{E}}) &= 2 \int_0^1 \frac{u \operatorname{Tr} K_{11} + (1-u)c_1(\bar{S}) - c_1(\bar{S})}{u} du \\ &= 2(\operatorname{Tr} K_{11} - c_1(\bar{S})). \end{aligned}$$

To calculate  $\tilde{c}_2(\bar{\mathcal{E}})$ , use the identity  $2c_2 = c_1^2 - p_2$ .  $\square$

Corollary 1 (b) agrees with an important calculation of Deligne's in [D], 10.1, which we now describe: Using the  $C^\infty$  splitting of  $\mathcal{E}$ , we can write the  $\bar{\partial}$  operator for  $E$  in matrix form:

$$\bar{\partial}_E = \begin{pmatrix} \bar{\partial}_S & \alpha \\ 0 & \bar{\partial}_Q \end{pmatrix}, \quad \text{for some } \alpha \in A^{0,1}(X, \operatorname{Hom}(Q, S)).$$

Let  $\alpha^* \in A^{1,0}(X, \operatorname{Hom}(S, Q))$  be the transpose of  $\alpha$ , defined using complex conjugation of forms and the metrics  $h_S$  and  $h_Q$ . If  $\nabla$  is the induced connection on  $\operatorname{Hom}(Q, S)$ , we can write

$$K_E = \left( \begin{array}{c|c} K_S - \frac{i}{2\pi} \alpha \alpha^* & \nabla^{1,0} \alpha \\ \hline -\nabla^{0,1} \alpha^* & K_Q - \frac{i}{2\pi} \alpha^* \alpha \end{array} \right).$$

Thus Corollary 1(b) implies that

$$\tilde{c}_2(\bar{\mathcal{E}}) = -\frac{1}{2\pi i} \operatorname{Tr}(\alpha \alpha^*) = \frac{1}{2\pi i} \operatorname{Tr}(\alpha^* \alpha),$$

and we have recovered Deligne's result. In this form the calculation was used by Moriwaki and Soulé to obtain a Bogomolov-Gieseker type inequality and a Kodaira vanishing theorem on arithmetic surfaces, respectively (see [Mo] and [S]).

The calculation of  $\tilde{c}_2$  shows that in general Bott-Chern forms are not closed. In fact, calculating  $\tilde{c}_k$  for  $k \geq 3$  leads to much more complicated formulas, involving traces of products of curvature matrices, for which a clear geometric interpretation is lacking (unlike the matrix  $\alpha$  above, whose negative transpose  $-\alpha^*$  is the second fundamental form of  $\bar{\mathcal{E}}$ ). In the next two sections we shall see that when  $\bar{E}$  is a projectively flat bundle, the Bott-Chern forms are closed and can be calculated explicitly for any  $\phi \in I(n)$ .