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**Autor:** Tamvakis, Harry  
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## 4. CALCULATING BOTT-CHERN FORMS

In this section we will consider an exact sequence

$$\bar{\mathcal{E}} : 0 \rightarrow \bar{S} \rightarrow \bar{E} \rightarrow \bar{Q} \rightarrow 0,$$

where the metrics on  $\bar{S}$  and  $\bar{Q}$  are induced from the metric on  $E$ . Let  $r, n$  be the ranks of the bundles  $S$  and  $E$ . Let  $\phi \in I(n)$  be homogeneous of degree  $k$ . We will formulate a theorem for calculating the Bott-Chern form  $\tilde{\phi}(\bar{\mathcal{E}})$ . This result follows from the work of Bott-Chern, Cowen, Bismut and Gillet-Soulé.

Let  $\phi'$  be defined as in §2. For any two matrices  $A, B \in M_n(\mathbb{C})$  set

$$\phi'(A; B) := \sum_{i=1}^k \phi'(A, A, \dots, A, B_{(i)}, A, \dots, A),$$

where the index  $i$  means that  $B$  is in the  $i$ -th position.

Choose a local orthonormal frame  $s = (s_1, s_2, \dots, s_n)$  of  $E$  such that the first  $r$  elements generate  $S$ , and let  $K(\bar{S})$ ,  $K(\bar{E})$  and  $K(\bar{Q})$  be the curvature matrices of  $\bar{S}$ ,  $\bar{Q}$  and  $\bar{E}$  with respect to  $s$ . Let  $K_S = \frac{i}{2\pi} K(\bar{S})$ ,  $K_E = \frac{i}{2\pi} K(\bar{E})$  and  $K_Q = \frac{i}{2\pi} K(\bar{Q})$ . The matrix  $K_E$  has the form

$$K_E = \left( \begin{array}{c|c} K_{11} & K_{12} \\ \hline K_{21} & K_{22} \end{array} \right)$$

where  $K_{11}$  is an  $r \times r$  submatrix. Also consider the matrices

$$K_0 = \left( \begin{array}{c|c} K_S & 0 \\ \hline K_{21} & K_Q \end{array} \right) \quad \text{and} \quad J_r = \left( \begin{array}{c|c} Id_r & 0 \\ \hline 0 & 0 \end{array} \right).$$

Let  $u$  be a formal variable and  $K(u) := uK_E + (1 - u)K_0$ . Finally, let  $\phi^!(u) = \phi'(K(u); J_r)$ . We then have the following

**THEOREM 2.**

$$(3) \quad \tilde{\phi}(\bar{\mathcal{E}}) = \int_0^1 \frac{\phi^!(u) - \phi^!(0)}{u} du.$$

*Proof.* We prove that  $\tilde{\phi}(\bar{\mathcal{E}})$  as defined above satisfies axioms (i)-(iii) of Theorem 1. The main step is the first axiom; this was essentially done in [BC] §4, when  $\phi = c$  is the total Chern class. In the form (3) (again for the total Chern class), the equation was given by Cowen in [C1] and [C2], while

simplifying Bott and Chern's proof. We follow both sources in sketching a proof of this more general result.

Let  $h$  and  $h_Q$  denote the metrics on  $E$  and  $Q$  respectively. Define the orthogonal projections  $P_1 : \bar{E} \rightarrow \bar{S}$  and  $P_2 : \bar{E} \rightarrow \bar{Q}$  and put  $h_u(v, v') = uh(P_1v, P_1v') + h(P_2v, P_2v')$  for  $v, v' \in E_x$  and  $0 < u \leq 1$ . Then  $h_u$  is a hermitian norm,  $h_1 = h$  and  $h_u \rightarrow h_Q$  as  $u \rightarrow 0$ . Let  $K(E, h_u)$  be the curvature matrix of  $(E, h_u)$  relative to the holomorphic frame  $s$  defined above. Proposition 3.1 of [C2] proves that  $\frac{i}{2\pi}K(E, h_u) = K(u)$ . It follows from Proposition 3.28 of [BC] that for  $0 < t \leq 1$ ,

$$\phi(E, h_t) - \phi(E, h) = dd^c \int_t^1 \frac{\phi'(K(u); J_r)}{u} du.$$

If we could set  $t = 0$  we would be done; however, the integral will not be convergent in general. Note that  $K(u) = K_0 + uK_1$ , where  $K_1 \in A^{1,1}(X, \text{End}(E))$  is independent of  $u$ . Therefore it will suffice to show that  $\phi'(K_0; J_r)$  is a closed form, so that it can be deleted from the integral. For this we may assume that  $\phi = p_\lambda$  is a product of power sums,  $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_m)$  a partition. Then

$$\begin{aligned} p'_\lambda(K_0; J_r) &= \sum_{i=1}^m \text{Tr}(K_S)^{\lambda_i-1} \prod_{j \neq i} (\text{Tr}(K_S) + \text{Tr}(K_Q))^{\lambda_j} \\ &= \sum_{i=1}^m p_{\lambda_i-1}(\bar{S}) \prod_{j \neq i} p_{\lambda_j}(\bar{S} \oplus \bar{Q}) \end{aligned}$$

is certainly a closed form.

This proves axioms (i) and (iii); axiom (ii) is easily checked as well.  $\square$

REMARK. A similar deformation to the one in [C2] was used by Deligne in [D], 5.11 for a calculation involving the Chern character form. Special cases of Theorem 2 have been used in the literature before, see for example [GS2] Prop. 5.3, [GSZ] 2.2.3 and [Ma] Theorem 3.3.1.

We deduce some simple but useful calculations:

COROLLARY 1.

- (a)  $\tilde{c}_1^k(\bar{\mathcal{E}}) = 0$  for all  $k \geq 1$  and  $\tilde{c}_m(\bar{\mathcal{E}}) = 0$  for all  $m > \text{rk } E$ .
- (b)  $\tilde{p}_2(\bar{\mathcal{E}}) = 2(\text{Tr } K_{11} - c_1(\bar{S}))$  and  $\tilde{c}_2(\bar{\mathcal{E}}) = c_1(\bar{S}) - \text{Tr } K_{11}$ .

*Proof.* (a)  $c_1^1(u)$  is independent of  $u$ ; hence  $\tilde{c}_1(\bar{\mathcal{E}}) = 0$ . The result for higher powers of  $c_1$  follows from Proposition 1. In addition,  $\tilde{c}_m(\bar{\mathcal{E}}) = 0$  for  $m > \text{rk } E$  is an immediate consequence of the definition.

(b) Using the bilinear form  $p'_2$  described previously, we find  $p'_2(u) = 2(u \operatorname{Tr} K_{11} + (1-u)c_1(\bar{S}))$ , so

$$\begin{aligned} \tilde{p}_2(\bar{\mathcal{E}}) &= 2 \int_0^1 \frac{u \operatorname{Tr} K_{11} + (1-u)c_1(\bar{S}) - c_1(\bar{S})}{u} du \\ &= 2(\operatorname{Tr} K_{11} - c_1(\bar{S})). \end{aligned}$$

To calculate  $\tilde{c}_2(\bar{\mathcal{E}})$ , use the identity  $2c_2 = c_1^2 - p_2$ .  $\square$

Corollary 1 (b) agrees with an important calculation of Deligne's in [D], 10.1, which we now describe: Using the  $C^\infty$  splitting of  $\mathcal{E}$ , we can write the  $\bar{\partial}$  operator for  $E$  in matrix form:

$$\bar{\partial}_E = \begin{pmatrix} \bar{\partial}_S & \alpha \\ 0 & \bar{\partial}_Q \end{pmatrix}, \quad \text{for some } \alpha \in A^{0,1}(X, \operatorname{Hom}(Q, S)).$$

Let  $\alpha^* \in A^{1,0}(X, \operatorname{Hom}(S, Q))$  be the transpose of  $\alpha$ , defined using complex conjugation of forms and the metrics  $h_S$  and  $h_Q$ . If  $\nabla$  is the induced connection on  $\operatorname{Hom}(Q, S)$ , we can write

$$K_E = \left( \begin{array}{c|c} K_S - \frac{i}{2\pi} \alpha \alpha^* & \nabla^{1,0} \alpha \\ \hline -\nabla^{0,1} \alpha^* & K_Q - \frac{i}{2\pi} \alpha^* \alpha \end{array} \right).$$

Thus Corollary 1(b) implies that

$$\tilde{c}_2(\bar{\mathcal{E}}) = -\frac{1}{2\pi i} \operatorname{Tr}(\alpha \alpha^*) = \frac{1}{2\pi i} \operatorname{Tr}(\alpha^* \alpha),$$

and we have recovered Deligne's result. In this form the calculation was used by Moriwaki and Soulé to obtain a Bogomolov-Gieseker type inequality and a Kodaira vanishing theorem on arithmetic surfaces, respectively (see [Mo] and [S]).

The calculation of  $\tilde{c}_2$  shows that in general Bott-Chern forms are not closed. In fact, calculating  $\tilde{c}_k$  for  $k \geq 3$  leads to much more complicated formulas, involving traces of products of curvature matrices, for which a clear geometric interpretation is lacking (unlike the matrix  $\alpha$  above, whose negative transpose  $-\alpha^*$  is the second fundamental form of  $\bar{\mathcal{E}}$ ). In the next two sections we shall see that when  $\bar{E}$  is a projectively flat bundle, the Bott-Chern forms are closed and can be calculated explicitly for any  $\phi \in I(n)$ .