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by the *power sum forms* in the results; to our knowledge this phenomenon has not been observed before. The combinatorial identities involving harmonic numbers that we encounter are also interesting. Sections 2-6 contain results in hermitian complex geometry and may be read without prior knowledge of Arakelov theory. §8 applies our calculations to obtain a presentation of the Arakelov Chow ring of the arithmetic Grassmannian.

This should be regarded as a companion paper to [T]; both papers will be part of the author's 1997 University of Chicago thesis. I wish to thank my advisor William Fulton for many useful conversations and exchanges of ideas.

## 2. INVARIANT AND SYMMETRIC FUNCTIONS

The symmetric group  $S_n$  acts on the polynomial ring  $\mathbf{Z}[x_1, x_2, \dots, x_n]$  by permuting the variables, and the ring of invariants  $\Lambda(n) = \mathbf{Z}[x_1, x_2, \dots, x_n]^{S_n}$  is the ring of symmetric polynomials. For  $B = \mathbf{Q}$  or  $\mathbf{C}$ , let  $\Lambda(n, B) = \Lambda(n) \otimes_{\mathbf{Z}} B$ .

Let  $e_k(x_1, \dots, x_n)$  be the  $k$ -th elementary symmetric polynomial in the variables  $x_1, \dots, x_n$  and  $p_k(x_1, \dots, x_n) = \sum_i x_i^k$  the  $k$ -th power sum. The fundamental theorem on symmetric functions states that  $\Lambda(n) = \mathbf{Z}[e_1, \dots, e_n]$  and that  $e_1, \dots, e_n$  are algebraically independent. For  $\lambda$  a partition, i.e. a decreasing sequence  $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_m$  of nonnegative integers, define  $p_\lambda := \prod_{i=1}^m p_{\lambda_i}$ . It is well known that the  $p_\lambda$ 's form an additive  $\mathbf{Q}$ -basis for the ring of symmetric polynomials (cf. [M], §I.2). The two bases are related by Newton's identity:

$$(2) \quad p_k - e_1 p_{k-1} + e_2 p_{k-2} - \dots + (-1)^k k e_k = 0.$$

Another important set of symmetric functions related to the cohomology ring of grassmannians are the Schur polynomials. For a partition  $\lambda$  as above, the Schur polynomial  $s_\lambda$  is defined by

$$s_\lambda(x_1, \dots, x_n) = \frac{1}{\Delta} \cdot \det(x_i^{\lambda_j + n - j})_{1 \leq i, j \leq n},$$

where  $\Delta = \prod_{1 \leq i < j \leq n} (x_i - x_j)$  is the Vandermonde determinant. The  $s_\lambda$  for all  $\lambda$  of length  $m \leq n$  form a  $\mathbf{Z}$ -basis of  $\Lambda(n)$  (cf. [M], §I.3).

Let  $\mathbf{C}[T_{ij}]$  ( $1 \leq i, j \leq n$ ) be the coordinate ring of the space  $M_n(\mathbf{C})$  of  $n \times n$  matrices.  $GL_n(\mathbf{C})$  acts on matrices by conjugation, and we let

$I(n) = \mathbf{C}[T_{ij}]^{GL_n(\mathbf{C})}$  denote the corresponding graded ring of invariants. There is an isomorphism  $\tau: I(n) \rightarrow \Lambda(n, \mathbf{C})$  given by evaluating an invariant polynomial  $\phi$  on the diagonal matrix  $\text{diag}(x_1, \dots, x_n)$ . We will often identify  $\phi$  with the symmetric polynomial  $\tau(\phi)$ . We will need to consider invariant polynomials with rational coefficients; let  $I(n, \mathbf{Q}) \simeq \mathbf{Q}[x_1, x_2, \dots, x_n]^{S_n}$  be the corresponding ring.

Given  $\phi \in I(n)_k$ , let  $\phi'$  be a  $k$ -multilinear form on  $M_n(\mathbf{C})$  such that

$$\phi'(gA_1g^{-1}, \dots, gA_kg^{-1}) = \phi'(A_1, \dots, A_k)$$

for  $g \in GL(n, \mathbf{C})$  and  $\phi(A) = \phi'(A, A, \dots, A)$ . Such forms are most easily constructed for the power sums  $p_k$  by setting

$$p'_k(A_1, A_2, \dots, A_k) = \text{Tr}(A_1A_2 \cdots A_k).$$

For  $p_\lambda$  we can take  $p'_\lambda = \prod p'_{\lambda_i}$ . Since the  $p_\lambda$ 's are a basis of  $\Lambda(n, \mathbf{Q})$ , it follows that one can use the above constructions to find multilinear forms  $\phi'$  for any  $\phi \in I(n)_k$ .

An explicit formula for  $\phi'$  is given by polarizing  $\phi$ :

$$\phi'(A_1, \dots, A_k) = \frac{(-1)^k}{k!} \sum_{j=1}^k \sum_{i_1 < \dots < i_j} (-1)^j \phi(A_{i_1} + \dots + A_{i_j}).$$

Although above formula for  $\phi'$  is symmetric in  $A_1, \dots, A_k$ , this property is not needed for the applications that follow.

### 3. HERMITIAN DIFFERENTIAL GEOMETRY

Let  $X$  be a complex manifold,  $E$  a rank  $n$  holomorphic vector bundle over  $X$ . Denote by  $A^k(X, E)$  the  $C^\infty$  sections of  $\Lambda^k T^*X \otimes E$ , where  $T^*X$  denotes the cotangent bundle of  $X$ . In particular  $A^k(X)$  is the space of smooth complex  $k$ -forms on  $X$ . Let  $A^{p,q}(X)$  the space of smooth complex forms of type  $(p, q)$  on  $X$  and  $A(X) := \bigoplus_p A^{p,p}(X)$ . The decomposition  $A^1(X, E) = A^{1,0}(X, E) \oplus A^{0,1}(X, E)$  induces a decomposition  $D = D^{1,0} + D^{0,1}$  of each connection  $D$  on  $E$ . Let  $d = \partial + \bar{\partial}$  and  $d^c = (\partial - \bar{\partial})/(4\pi i)$ .

Assume now that  $E$  is equipped with a hermitian metric  $h$ . The pair  $(E, h)$  is called a *hermitian vector bundle*. The metric  $h$  induces a canonical connection  $D = D(h)$  such that  $D^{0,1} = \bar{\partial}_E$  and  $D$  is *unitary*, i.e.

$$dh(s, t) = h(Ds, t) + h(s, Dt), \quad \text{for all } s, t \in A^0(X, E).$$