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**Autor:** Ceccherini-Silberstein, Tullio G. / GRIGORCHUK, Rostislav I.  
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3.6. THEOREM ([L 1,2]). *Let  $R$  be an element of the free group  $F$  of finite rank  $m$  which is primitive with respect to the lower central series. Denote by  $k = \omega(R)$  its weight and by  $\langle R \rangle$  the normal closure of  $R$  in  $F$ . Let  $G = F/\langle R \rangle$  and let  $\mathcal{L}(F)$  and  $\mathcal{L}(G)$  be the corresponding Lie algebras. Let then  $r$  be the image of  $R$  in  $\mathcal{L}_k(F)$ , the  $k$ -th component of  $\mathcal{L}(F)$  and denote by  $I$  the ideal of  $\mathcal{L}(F)$  generated by  $r$ .*

*Then  $I$  is the kernel of the canonical homomorphism of  $\mathcal{L}(F)$  onto  $\mathcal{L}(G)$ , i.e.*

$$\mathcal{L}(G) = \mathcal{L}(F)/I.$$

*Moreover for all  $n \geq 1$  the abelian group  $\mathcal{L}_n(G)$  is a torsion free group whose rank is the  $n$ -th coefficient of the Maclaurin power series of the function*

$$U(z) = \frac{1}{1 - mz + z^k}.$$

#### 4. MORE ON UNIFORMLY EXPONENTIAL GROWTH OF ONE-RELATOR GROUPS

Any two-generated one-relator group  $G$  can be presented in the form  $G = \langle a, b : a^k w(a, b) = 1 \rangle$  where  $k \in \mathbf{Z}$  and  $w(a, b)$  belongs to the commutator subgroup  $[F, F]$  of the free group  $F = F(a, b)$  freely generated by  $a$  and  $b$  (this follows from Lemma 1.1). Since  $a$  and  $b$  constitute a basis in  $F/\gamma_2(F)$  and  $[a, b]$  generates  $\gamma_2(F)/\gamma_3(F)$ , one can also present  $G$  in the form

$$G = \langle a, b : a^k [a, b]^l w(a, b) = 1 \rangle$$

where  $k, l \in \mathbf{Z}$  and  $w(a, b) \in \gamma_3(F)$ .

In this section we shall see that, under suitable assumptions on  $k, l$  and  $w(a, b)$ , the corresponding group has uniformly exponential growth.

As an application of Labute's Theorem we get the following:

4.1. PROPOSITION. *Let  $G = \langle a, b : R(a, b) = 1 \rangle$  be such that  $R$  is primitive with respect to  $\{\gamma_n(F)\}_{n=1}^\infty$  and  $R \in \gamma_3(F)$ . Then  $G$  has uniformly exponential growth.*

*Proof.* If  $\omega(R) \geq 3$ , Theorem 3.6 shows that the corresponding function  $U(z)$  has a pole  $z_0$  with  $0 < z_0 < 1$ . It follows that the coefficients  $c_n(G)$  grow exponentially. By Corollary 3.2,  $\lambda_*(G) > 1$ .  $\square$

For Proposition 4.3 we need the following notations. Let  $\xi$  be a positive rational number such that  $\xi \neq 1$  and denote by  $Q_\xi$  the smallest subgroup of the additive group of the rationals, which contains 1 and is invariant under multiplication by  $\xi$  and  $\xi^{-1}$ . In other words if  $\xi = \frac{p}{q}$  with  $p, q \in \mathbf{Z}$  and  $\gcd(p, q) = 1$  then  $Q_\xi \equiv \mathbf{Z}[\frac{1}{p}, \frac{1}{q}]$ . Consider now the automorphism  $\alpha$  of  $Q_\xi$  defined by  $\alpha(x) = \xi x$ ,  $x \in Q_\xi$ . Let  $\mathbf{Z}$  act on  $Q_\xi$  by powers of  $\alpha$ . Denote by  $G_\xi = Q_\xi \rtimes_\alpha \mathbf{Z}$  the corresponding semidirect product. The group  $G_\xi$  is a two-generated group with system of generators  $\{\bar{a}, \bar{b}\}$ , where  $\bar{a} = 1 \in Q_\xi$  and the element  $\bar{b}$  implements the automorphism  $\alpha: \bar{b}^{-1}x\bar{b} = \alpha(x)$ ,  $x \in Q_\xi$ .

Let now  $d$  be a natural number  $\geq 2$  and set  $B_d = \prod_{\mathbf{Z}} \mathbf{Z}_d$ . The group  $\mathbf{Z}$  acts on  $B_d$  by shifts. The corresponding semidirect product  $\Gamma(d)$ , also denoted by  $\mathbf{Z}_d \wr \mathbf{Z}$ , is called the *wreath product* of  $\mathbf{Z}$  and  $\mathbf{Z}_d$ . We shall consider  $\Gamma(d)$  as generated by  $\bar{a} = (\dots, 0, 0, 1, 0, 0, \dots)$  where 1 denotes a generator of  $\mathbf{Z}_d$  (in the expression of  $\bar{a}$  it appears at the 0-th coordinate place), and by  $\bar{b}$ , the element which implements the shift.

We have short exact sequences

$$\begin{aligned} 0 &\longrightarrow Q_\xi \longrightarrow G_\xi \longrightarrow \mathbf{Z} \longrightarrow 0 \\ 0 &\longrightarrow B_d \longrightarrow \Gamma(d) \longrightarrow \mathbf{Z} \longrightarrow 0 \end{aligned}$$

so that  $G_\xi$  and  $\Gamma(d)$  are two-step solvable. Slightly modifying the proof of Proposition 2.6 one gets

4.2. LEMMA. *The groups  $G_\xi$  and  $\Gamma(d)$  have uniformly exponential growth.*

Our last class of two-generated one-relator groups of uniformly exponential growth is determined in the following statement.

4.3. PROPOSITION. *Let  $G = \langle a, b; a^k[a, b]^l w(a, b) = 1 \rangle$  with  $k, l \in \mathbf{Z}$  and  $w(a, b) \in F^{(2)}$  where  $F^{(2)} = [[F, F], [F, F]]$  denotes the second commutator subgroup of the free group  $F = F(a, b)$  on  $a$  and  $b$ . Suppose that  $(k, l) \notin \{\pm(2, 1), \pm(1, 1), \pm(1, 0), \pm(0, 1)\}$ . Then  $G$  has uniformly exponential growth.*

*Proof.* Set  $G_{k,l} = \langle a, b; a^k[a, b]^l w(a, b) \rangle$ . Set also

$$\bar{G}_{k,l} = \langle a, b; a^k[a, b]^l w(a, b), F^{(2)} \rangle = \langle a, b; a^k[a, b]^l, F^{(2)} \rangle$$

which is a 2-step solvable quotient group of  $G_{k,l}$ . We shall show that  $\bar{G}_{k,l}$  can be mapped homomorphically onto either  $G_\xi$  or  $\Gamma(d)$  for a suitable positive rational number  $\xi \neq 1$  or natural number  $d \geq 2$ .

Suppose first that  $k \neq l, 2l$  and  $lk \neq 0$ . These assumptions guarantee that  $\xi := \left| \frac{l-k}{l} \right| \neq 0, 1$ . Then the map  $a \mapsto (\bar{a})^{\text{sgn}(\frac{l-k}{l})}, b \mapsto \bar{b}$  from  $F$  onto  $G_\xi$  factorizes through  $\bar{G}_{k,l}$ . Indeed if we suppose, for instance, that  $\frac{l-k}{l} > 0$ , then the image of  $a^k[a, b]^l$  is the number  $k + l(-1 + \xi) \in \mathbf{Q}_\xi$  which is zero. Thus  $\bar{G}_{k,l}$  maps onto  $G_\xi$ .

Suppose now that  $\gcd(k, l) = d$  or  $(k, l) \in \{\pm(d, 0), \pm(0, d)\}$  for some  $d \geq 2$ . Then, the same arguments as before show that  $\bar{G}_{k,l}$  can be mapped onto  $\Gamma(d)$  via the map  $a \mapsto \bar{a}, b \mapsto \bar{b}$ .

Finally observe that  $\bar{G}_{0,0}$  is the free two-generated two-step solvable group  $F/F^{(2)}$  and thus maps homomorphically onto  $\Gamma(d)$  for any  $d \geq 2$ .

The proof follows from Lemma 4.2.  $\square$

Remark that the two-generated one-relator groups that are not covered by our statements have their relator that can be reduced to one of the form  $bw$ ,  $[a, b]w$  or  $ba^{-1}baw$ , where  $w = w(a, b) \in F^{(2)}$ .

Let us finish the paper by the following observation.

In [GrLP] it was conjectured that if  $G$  is a group with  $m$  generators and  $p$  relations, then

$$\lambda_*(G) \geq 2(m - p) - 1.$$

For one-relator groups there is one case when Gromov's conjecture holds true.

**4.4. PROPOSITION.** *Let  $G = \langle a_1, a_2, \dots, a_m : R(a_1, a_2, \dots, a_m) = 1 \rangle$ , with  $m \geq 2$ , be a one-relator group such that the relator  $R$  does not belong to the commutator subgroup  $F'$  of the free group  $F$  of rank  $m$  freely generated by  $a_1, a_2, \dots, a_m$ . Then  $\lambda_*(G) \geq 2m - 3$ .*

*Proof.* We may assume that  $G$  is torsion-free. Indeed if  $U, V \in F$  are such  $U = V^k$  for some  $k \in \mathbf{Z}$ , then  $U \in F'$  iff  $V \in F'$ . If the relator  $R$  is a proper power, say  $R = W^k$ , then  $G$  maps onto  $G_1 = \langle a_1, a_2, \dots, a_m : W(a_1, a_2, \dots, a_m) = 1 \rangle$ , which is torsion-free, and  $\lambda_*(G) \geq \lambda_*(G_1)$ .

Under our assumptions on  $R$ ,  $H_1(G, \mathbf{Q}) \cong \mathbf{Z}^{m-1}$  and the second rational homology group  $H_2(G, \mathbf{Q})$  vanishes.

In [S] it is proven that if  $H_2(G, \mathbf{k}) = 0$ , where  $\mathbf{k}$  is a field, then any subset  $\{x_j\} \in G$ , whose image in  $H_1(G, \mathbf{k})$  is linearly independent, freely generates a free group.

Let  $X = \{x_1, x_2, \dots, x_n\}$  be a finite system of generators for  $G$ . Then  $\bar{X} = \{\bar{x}_1, \dots, \bar{x}_n\}$ , where  $\bar{x}_i$  denotes the image of  $x_i$  in  $H_1(G, \mathbf{Q})$ , generates

$H_1(G, \mathbf{Q})$ . We can find an independent subsystem  $\{\bar{x}_{i_1}, \dots, \bar{x}_{i_{m-1}}\}$  in  $H_1(G, \mathbf{Q})$  such that its pre-image  $\{x_{i_1}, \dots, x_{i_m}\}$  freely generates a free group. Therefore  $\lambda_X(G) \geq 2(m-1) - 1 = 2m - 3$ .  $\square$

It seems to us that for a one-relator group  $G$  of rank  $m \geq 3$  the inequality  $\lambda_*(G) \geq 2m - 3$  cannot be deduced directly from Magnus' Theorem as it is claimed in [GrLP].

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