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3. UNIFORMLY EXPONENTIAL GROWTH AND GROWTH OF GRADED ALGEBRAS

In this section we describe a method of estimating growth functions of a group in terms of its graded Lie, and associative algebras defined via dimension subgroups. We begin by recalling some concepts and notations.

As in [Gri] considerations were given with respect to a Galois field \mathbf{GF}_p , here we modify the arguments for a field of characteristic 0, namely \mathbf{Q} .

Let G be a group; denote by $\mathbf{Q}[G]$ the group algebra of G over \mathbf{Q} , and by $\Delta \subset \mathbf{Q}[G]$ the augmentation ideal, that is the ideal generated by the elements of the form $g - 1$, with $g \in G$. Recall that the *lower central series* of G is the sequence of subgroups $\{\gamma_n(G)\}_{n=1}^{\infty}$ of G defined by $\gamma_1(G) = G$ and, for $n \geq 2$, $\gamma_n(G) = [G, \gamma_{n-1}(G)]$.

The subgroup

$$G_n = \{g \in G : g - 1 \in \Delta^n\}$$

is called the *n-th dimension subgroup* of G over \mathbf{Q} and it has the following characterisation due to Jennings [J] (see also [P: IV, Thm. 1.5] or [Pm: 11, Thm. 1.10])

$$G_n = \sqrt{\gamma_n(G)} := \{g \in G : \exists k \in \mathbf{N}, g^k \in \gamma_n(G)\}.$$

For any group G one defines as usual an associative graded algebra $\mathcal{A}(G)$ and two graded Lie algebras $L(G)$ and $\mathcal{L}(G)$ by

$$\begin{aligned} \mathcal{A}(G) &= \bigoplus_{n=1}^{\infty} \Delta^n / \Delta^{n+1} \\ L(G) &= \bigoplus_{n=1}^{\infty} [(G_n / G_{n+1}) \otimes_{\mathbf{Z}} \mathbf{Q}] \\ \mathcal{L}(G) &= \bigoplus_{n=1}^{\infty} [(\gamma_n(G) / \gamma_{n+1}(G)) \otimes_{\mathbf{Z}} \mathbf{Q}] \end{aligned}$$

(see for instance [P], [Pm]). Quillen's Theorem [Q] states that $\mathcal{A}(G)$ is the universal enveloping algebra of $L(G)$.

Assume now that G is finitely generated and set

$$\begin{aligned} a_n(G) &= \dim(\Delta^n / \Delta^{n+1}) \\ b_n(G) &= \text{rank}(G_n / G_{n+1}) \\ c_n(G) &= \text{rank}(\gamma_n(G) / \gamma_{n+1}(G)) \end{aligned}$$

where, by rank, we mean the torsion free rank of the corresponding abelian group. Then the following relations hold

$$\sum_{n=0}^{\infty} a_n(G)z^n = \prod_{n=1}^{\infty} (1 - z^n)^{-b_n(G)} = \prod_{n=1}^{\infty} (1 - z^n)^{-c_n(G)}.$$

The first equality follows easily from Quillen's Theorem [Pm: Thm. 4.10, Chapter 3] and the second one follows from the equality $b_n(G) = c_n(G)$ as proved in [Be].

In [Be] it is also proved that

$$\limsup_{n \rightarrow \infty} \sqrt[n]{a_n} = \limsup_{n \rightarrow \infty} \sqrt[n]{c_n}.$$

3.1. LEMMA. *For any finite system of generators A of a group G the following inequality holds:*

$$a_n(G) \leq \gamma_A^G(n), \quad n \geq 1.$$

Proof. For $x, y \in G$ we have

$$\begin{aligned} xy - 1 &= (x - 1) + (y - 1) + (x - 1)(y - 1) \\ x^{-1} - 1 &= -(x - 1) - (x - 1)(x^{-1} - 1) \end{aligned}$$

so that

$$\begin{aligned} xy - 1 &\equiv (x - 1) + (y - 1) \pmod{\Delta^2} \\ x^{-1} - 1 &\equiv -(x - 1) \pmod{\Delta^2}. \end{aligned}$$

The ideal Δ^n is spanned, over \mathbf{Q} , by the elements of the form

$$y_1(x_1 - 1)y_2(x_2 - 1) \cdots y_n(x_n - 1)y_{n+1},$$

where $x_i \in G$ and $y_j \in \mathbf{Q}[G]$, $1 \leq i \leq n$, $1 \leq j \leq n+1$. Since

$$y = \sum_{g \in G} k_g g \equiv \sum_{g \in G} k_g \pmod{\Delta}, \quad k_g \in \mathbf{Q}$$

a basis for the quotient space Δ^n/Δ^{n+1} can be chosen among the images modulo Δ^{n+1} of the elements of the form

$$(a_{i_1} - 1)(a_{i_2} - 1) \cdots (a_{i_n} - 1),$$

where $a_{i_j} \in A$. But $(a_{i_1} - 1)(a_{i_2} - 1) \cdots (a_{i_n} - 1) = \sum_{g \in G} k'_g g$, where the summation extends over elements g of length at most n with respect to the system of generators A . \square

3.2. COROLLARY. *Let G be a finitely generated group and suppose that the ranks of $\gamma_n(G)/\gamma_{n+1}(G)$ grow exponentially. Then G has uniformly exponential growth and the estimate*

$$\lambda_*(G) \geq \limsup_{n \rightarrow \infty} \sqrt[n]{\text{rank}(\gamma_n(G)/\gamma_{n+1}(G))}$$

holds.

Recall that a group G is *parafree of para-rank m* if it is residually nilpotent and the factors of consecutive groups in its lower central series equal the corresponding ones of a free group of rank m . There are parafree groups which are not isomorphic to free groups [B 2,3].

3.3. PROPOSITION. *A finitely generated parafree group G of para-rank $m \geq 2$ has uniformly exponential growth and*

$$\lambda_*(G) \geq m.$$

Proof. It is known (see for instance [MKS : Thms. 5.11 (Witt's Formulae) and 5.12]) that for a free group \mathbf{F}_m the rank of $(\gamma_n(\mathbf{F}_m)/\gamma_{n+1}(\mathbf{F}_m))$ equals the n -th coefficient of the Maclaurin power series of the function $U(z) = 1/(1-mz)$ and the previous corollary can be applied. \square

Given a parafree group G of para-rank $m \geq 2$ it would be interesting to compare $\lambda_*(G)$ with $\lambda_*(\mathbf{F}_m) = 2m - 1$.

3.4. PROBLEM. *Is it true that, for a finitely generated para-free group G of para-rank $m \geq 2$ which is not free, one has $\lambda_*(G) > 2m - 1$?*

In order to formulate the next statement we recall the following

3.5. DEFINITION. An element $R \in F$ is said to be *primitive with respect to the lower central series* if, for all $n \geq 2$, it is not an n -th power modulo $\gamma_{\omega(R)+1}(F)$ where $\omega(R)$ is the weight of R . (The latter is defined by $R \in \gamma_{\omega(R)}(F)$ but $R \notin \gamma_{\omega(R)+1}(F)$.)

3.6. THEOREM ([L 1,2]). *Let R be an element of the free group F of finite rank m which is primitive with respect to the lower central series. Denote by $k = \omega(R)$ its weight and by $\langle R \rangle$ the normal closure of R in F . Let $G = F/\langle R \rangle$ and let $\mathcal{L}(F)$ and $\mathcal{L}(G)$ be the corresponding Lie algebras. Let then r be the image of R in $\mathcal{L}_k(F)$, the k -th component of $\mathcal{L}(F)$ and denote by I the ideal of $\mathcal{L}(F)$ generated by r .*

Then I is the kernel of the canonical homomorphism of $\mathcal{L}(F)$ onto $\mathcal{L}(G)$, i.e.

$$\mathcal{L}(G) = \mathcal{L}(F)/I.$$

Moreover for all $n \geq 1$ the abelian group $\mathcal{L}_n(G)$ is a torsion free group whose rank is the n -th coefficient of the Maclaurin power series of the function

$$U(z) = \frac{1}{1 - mz + z^k}.$$

4. MORE ON UNIFORMLY EXPONENTIAL GROWTH OF ONE-RELATOR GROUPS

Any two-generated one-relator group G can be presented in the form $G = \langle a, b : a^k w(a, b) = 1 \rangle$ where $k \in \mathbf{Z}$ and $w(a, b)$ belongs to the commutator subgroup $[F, F]$ of the free group $F = F(a, b)$ freely generated by a and b (this follows from Lemma 1.1). Since a and b constitute a basis in $F/\gamma_2(F)$ and $[a, b]$ generates $\gamma_2(F)/\gamma_3(F)$, one can also present G in the form

$$G = \langle a, b : a^k [a, b]^l w(a, b) = 1 \rangle$$

where $k, l \in \mathbf{Z}$ and $w(a, b) \in \gamma_3(F)$.

In this section we shall see that, under suitable assumptions on k, l and $w(a, b)$, the corresponding group has uniformly exponential growth.

As an application of Labute's Theorem we get the following:

4.1. PROPOSITION. *Let $G = \langle a, b : R(a, b) = 1 \rangle$ be such that R is primitive with respect to $\{\gamma_n(F)\}_{n=1}^{\infty}$ and $R \in \gamma_3(F)$. Then G has uniformly exponential growth.*

Proof. If $\omega(R) \geq 3$, Theorem 3.6 shows that the corresponding function $U(z)$ has a pole z_0 with $0 < z_0 < 1$. It follows that the coefficients $c_n(G)$ grow exponentially. By Corollary 3.2, $\lambda_*(G) > 1$. \square