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# 3. UNIFORMLY EXPONENTIAL GROWTH AND GROWTH OF GRADED ALGEBRAS

In this section we describe a method of estimating growth functions of a group in terms of its graded Lie, and associative algebras defined via dimension subgroups. We begin by recalling some concepts and notations.

As in [Gri] considerations were given with respect to a Galois field  $\mathbf{GF}_p$ , here we modify the arguments for a field of characteristic 0, namely  $\mathbf{Q}$ .

Let G be a group; denote by  $\mathbf{Q}[G]$  the group algebra of G over  $\mathbf{Q}$ , and by  $\Delta \subset \mathbf{Q}[G]$  the augmentation ideal, that is the ideal generated by the elements of the form g - 1, with  $g \in G$ . Recall that the *lower central series* of G is the sequence of subgroups  $\{\gamma_n(G)\}_{n=1}^{\infty}$  of G defined by  $\gamma_1(G) = G$ and, for  $n \ge 2$ ,  $\gamma_n(G) = [G, \gamma_{n-1}(G)]$ .

The subgroup

$$G_n = \{g \in G : g - 1 \in \Delta^n\}$$

is called the *n*-th dimension subgroup of G over  $\mathbf{Q}$  and it has the following characterisation due to Jennings [J] (see also [P: IV, Thm. 1.5] or [Pm: 11, Thm. 1.10])

$$G_n = \sqrt{\gamma_n(G)} := \left\{ g \in G : \exists k \in \mathbf{N}, g^k \in \gamma_n(G) \right\}.$$

For any group G one defines as usual an associative graded algebra  $\mathcal{A}(G)$ and two graded Lie algebras L(G) and  $\mathcal{L}(G)$  by

$$\mathcal{A}(G) = \bigoplus_{n=1}^{\infty} \Delta^n / \Delta^{n+1}$$
$$L(G) = \bigoplus_{n=1}^{\infty} \left[ (G_n / G_{n+1}) \otimes_{\mathbf{Z}} \mathbf{Q} \right]$$
$$\mathcal{L}(G) = \bigoplus_{n=1}^{\infty} \left[ (\gamma_n(G) / \gamma_{n+1}(G)) \otimes_{\mathbf{Z}} \mathbf{Q} \right]$$

(see for instance [P], [Pm]). Quillen's Theorem [Q] states that  $\mathcal{A}(G)$  is the universal enveloping algebra of L(G).

Assume now that G is finitely generated and set

$$a_n(G) = \dim(\Delta^n / \Delta^{n+1})$$
  

$$b_n(G) = \operatorname{rank}(G_n / G_{n+1})$$
  

$$c_n(G) = \operatorname{rank}(\gamma_n(G) / \gamma_{n+1}(G))$$

where, by rank, we mean the torsion free rank of the corresponding abelian group. Then the following relations hold

$$\sum_{n=0}^{\infty} a_n(G) z^n = \prod_{n=1}^{\infty} (1-z^n)^{-b_n(G)} = \prod_{n=1}^{\infty} (1-z^n)^{-c_n(G)}.$$

The first equality follows easily from Quillen's Theorem [Pm: Thm. 4.10, Chapter 3] and the second one follows from the equality  $b_n(G) = c_n(G)$  as proved in [Be].

In [Be] it is also proved that

$$\limsup_{n \longrightarrow \infty} \sqrt[n]{a_n} = \limsup_{n \longrightarrow \infty} \sqrt[n]{c_n}.$$

3.1. LEMMA. For any finite system of generators A of a group G the following inequality holds:

$$a_n(G) \leq \gamma_A^G(n), \quad n \geq 1.$$

*Proof.* For  $x, y \in G$  we have

$$xy - 1 = (x - 1) + (y - 1) + (x - 1)(y - 1)$$
$$x^{-1} - 1 = -(x - 1) - (x - 1)(x^{-1} - 1)$$

so that

$$xy - 1 \equiv (x - 1) + (y - 1) \mod \Delta^2$$
  
 $x^{-1} - 1 \equiv -(x - 1) \mod \Delta^2$ .

The ideal  $\Delta^n$  is spanned, over **Q**, by the elements of the form

$$y_1(x_1-1)y_2(x_2-1)\cdots y_n(x_n-1)y_{n+1}$$
,

where  $x_i \in G$  and  $y_j \in \mathbf{Q}[G]$ ,  $1 \le i \le n$ ,  $1 \le j \le n+1$ . Since

$$y = \sum_{g \in G} k_g g \equiv \sum_{g \in G} k_g \mod \Delta, \ k_g \in \mathbf{Q}$$

a basis for the quotient space  $\Delta^n/\Delta^{n+1}$  can be chosen among the images modulo  $\Delta^{n+1}$  of the elements of the form

$$(a_{i_1}-1)(a_{i_2}-1)\cdots(a_{i_n}-1),$$

where  $a_{i_j} \in A$ . But  $(a_{i_1} - 1)(a_{i_2} - 1) \cdots (a_{i_n} - 1) = \sum_{g \in G} k'_g g$ , where the summation extends over elements g of length at most n with respect to the system of generators A.

3.2. COROLLARY. Let G be a finitely generated group and suppose that the ranks of  $\gamma_n(G)/\gamma_{n+1}(G)$  grow exponentially. Then G has uniformly exponential growth and the estimate

$$\lambda_*(G) \ge \limsup_{n \longrightarrow \infty} \sqrt[n]{\operatorname{rank}(\gamma_n(G)/\gamma_{n+1}(G))}$$

holds.

Recall that a group G is *parafree of para-rank* m if it is residually nilpotent and the factors of consecutive groups in its lower central series equal the corresponding ones of a free group of rank m. There are parafree groups which are not isomorphic to free groups [B 2,3].

3.3. PROPOSITION. A finitely generated parafree group G of para-rank  $m \ge 2$  has uniformly exponential growth and

$$\lambda_*(G) \ge m$$
.

*Proof.* It is known (see for instance [MKS: Thms. 5.11 (Witt's Formulae) and 5.12]) that for a free group  $\mathbf{F}_m$  the rank of  $(\gamma_n(\mathbf{F}_m)/\gamma_{n+1}(\mathbf{F}_m))$  equals the *n*-th coefficient of the Maclaurin power series of the function U(z) = 1/(1-mz) and the previous corollary can be applied.  $\Box$ 

Given a parafree group G of para-rank  $m \ge 2$  it would be interesting to compare  $\lambda_*(G)$  with  $\lambda_*(\mathbf{F}_m) = 2m - 1$ .

3.4. PROBLEM. Is it true that, for a finitely generated para-free group G of para-rank  $m \ge 2$  which is not free, one has  $\lambda_*(G) > 2m - 1$ ?

In order to formulate the next statement we recall the following

3.5. DEFINITION. An element  $R \in F$  is said to be *primitive with respect* to the lower central series if, for all  $n \ge 2$ , it is not an *n*-th power modulo  $\gamma_{\omega(R)+1}(F)$  where  $\omega(R)$  is the weight of R. (The latter is defined by  $R \in \gamma_{\omega(R)}(F)$  but  $R \notin \gamma_{\omega(R)+1}(F)$ .) 3.6. THEOREM ([L 1,2]). Let R be an element of the free group F of finite rank m which is primitive with respect to the lower central series. Denote by  $k = \omega(R)$  its weight and by  $\langle R \rangle$  the normal closure of R in F. Let  $G = F/\langle R \rangle$  and let  $\mathcal{L}(F)$  and  $\mathcal{L}(G)$  be the corresponding Lie algebras. Let then r be the image of R in  $\mathcal{L}_k(F)$ , the k-th component of  $\mathcal{L}(F)$  and denote by I the ideal of  $\mathcal{L}(F)$  generated by r.

Then I is the kernel of the canonical homomorphism of  $\mathcal{L}(F)$  onto  $\mathcal{L}(G)$ , i.e.

$$\mathcal{L}(G) = \mathcal{L}(F)/I.$$

Moreover for all  $n \ge 1$  the abelian group  $\mathcal{L}_n(G)$  is a torsion free group whose rank is the n-th coefficient of the Maclaurin power series of the function

$$U(z) = \frac{1}{1 - mz + z^k} \cdot$$

# 4. MORE ON UNIFORMLY EXPONENTIAL GROWTH OF ONE-RELATOR GROUPS

Any two-generated one-relator group G can be presented in the form  $G = \langle a, b : a^k w(a, b) = 1 \rangle$  where  $k \in \mathbb{Z}$  and w(a, b) belongs to the commutator subgroup [F, F] of the free group F = F(a, b) freely generated by a and b (this follows from Lemma 1.1). Since a and b constitute a basis in  $F/\gamma_2(F)$  and [a, b] generates  $\gamma_2(F)/\gamma_3(F)$ , one can also present G in the form

$$G = \left\langle a, b : a^k[a, b]^l w(a, b) = 1 \right\rangle$$

where  $k, l \in \mathbb{Z}$  and  $w(a, b) \in \gamma_3(F)$ .

In this section we shall see that, under suitable assumptions on k.l and w(a,b), the corresponding group has uniformly exponential growth.

As an application of Labute's Theorem we get the following:

4.1. PROPOSITION. Let  $G = \langle a, b : R(a, b) = 1 \rangle$  be such that R is primitive with respect to  $\{\gamma_n(F)\}_{n=1}^{\infty}$  and  $R \in \gamma_3(F)$ . Then G has uniformly exponential growth.

*Proof.* If  $\omega(R) \ge 3$ , Theorem 3.6 shows that the corresponding function U(z) has a pole  $z_0$  with  $0 < z_0 < 1$ . It follows that the coefficients  $c_n(G)$  grow exponentially. By Corollary 3.2,  $\lambda_*(G) > 1$ .