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3. Uniformly exponential growth and growth of graded algebras

In this section we describe a method of estimating growth functions of a group in terms of its graded Lie, and associative algebras defined via dimension subgroups. We begin by recalling some concepts and notations.

As in [Gri] considerations were given with respect to a Galois field \mathbf{GF}_p , here we modify the arguments for a field of characteristic 0, namely \mathbf{Q} .

Let G be a group; denote by $\mathbb{Q}[G]$ the group algebra of G over \mathbb{Q} , and by $\Delta \subset \mathbb{Q}[G]$ the augmentation ideal, that is the ideal generated by the elements of the form g-1, with $g \in G$. Recall that the *lower central series* of G is the sequence of subgroups $\{\gamma_n(G)\}_{n=1}^{\infty}$ of G defined by $\gamma_1(G) = G$ and, for $n \geq 2$, $\gamma_n(G) = [G, \gamma_{n-1}(G)]$.

The subgroup

$$G_n = \{g \in G : g - 1 \in \Delta^n\}$$

is called the n-th dimension subgroup of G over \mathbb{Q} and it has the following characterisation due to Jennings [J] (see also [P: IV, Thm. 1.5] or [Pm: 11, Thm. 1.10])

$$G_n = \sqrt{\gamma_n(G)} := \{g \in G : \exists k \in \mathbb{N}, g^k \in \gamma_n(G)\}.$$

For any group G one defines as usual an associative graded algebra $\mathcal{A}(G)$ and two graded Lie algebras $\mathcal{L}(G)$ and $\mathcal{L}(G)$ by

$$\mathcal{A}(G) = \bigoplus_{n=1}^{\infty} \Delta^{n} / \Delta^{n+1}$$

$$L(G) = \bigoplus_{n=1}^{\infty} \left[(G_{n} / G_{n+1}) \otimes_{\mathbf{Z}} \mathbf{Q} \right]$$

$$\mathcal{L}(G) = \bigoplus_{n=1}^{\infty} \left[(\gamma_{n} (G) / \gamma_{n+1} (G)) \otimes_{\mathbf{Z}} \mathbf{Q} \right]$$

(see for instance [P], [Pm]). Quillen's Theorem [Q] states that $\mathcal{A}(G)$ is the universal enveloping algebra of L(G).

Assume now that G is finitely generated and set

$$a_n(G) = \dim(\Delta^n/\Delta^{n+1})$$

$$b_n(G) = \operatorname{rank}(G_n/G_{n+1})$$

$$c_n(G) = \operatorname{rank}(\gamma_n(G)/\gamma_{n+1}(G))$$

where, by rank, we mean the torsion free rank of the corresponding abelian group. Then the following relations hold

$$\sum_{n=0}^{\infty} a_n(G) z^n = \prod_{n=1}^{\infty} (1 - z^n)^{-b_n(G)} = \prod_{n=1}^{\infty} (1 - z^n)^{-c_n(G)}.$$

The first equality follows easily from Quillen's Theorem [Pm: Thm. 4.10, Chapter 3] and the second one follows from the equality $b_n(G) = c_n(G)$ as proved in [Be].

In [Be] it is also proved that

$$\limsup_{n \to \infty} \sqrt[n]{a_n} = \limsup_{n \to \infty} \sqrt[n]{c_n}.$$

3.1. Lemma. For any finite system of generators A of a group G the following inequality holds:

$$a_n(G) \le \gamma_A^G(n), \quad n \ge 1.$$

Proof. For $x, y \in G$ we have

$$xy - 1 = (x - 1) + (y - 1) + (x - 1)(y - 1)$$
$$x^{-1} - 1 = -(x - 1) - (x - 1)(x^{-1} - 1)$$

so that

$$xy - 1 \equiv (x - 1) + (y - 1) \mod \Delta^2$$

 $x^{-1} - 1 \equiv -(x - 1) \mod \Delta^2$.

The ideal Δ^n is spanned, over \mathbf{Q} , by the elements of the form

$$y_1(x_1-1)y_2(x_2-1)\cdots y_n(x_n-1)y_{n+1}$$
.

where $x_i \in G$ and $y_j \in \mathbf{Q}[G]$, $1 \le i \le n$, $1 \le j \le n+1$. Since

$$y = \sum_{g \in G} k_g g \equiv \sum_{g \in G} k_g \mod \Delta, \ k_g \in \mathbf{Q}$$

a basis for the quotient space Δ^n/Δ^{n+1} can be chosen among the images modulo Δ^{n+1} of the elements of the form

$$(a_{i_1}-1)(a_{i_2}-1)\cdots(a_{i_n}-1)$$
,

where $a_{i_j} \in A$. But $(a_{i_1} - 1)(a_{i_2} - 1) \cdots (a_{i_n} - 1) = \sum_{g \in G} k'_g g$, where the summation extends over elements g of length at most n with respect to the system of generators A.

3.2. COROLLARY. Let G be a finitely generated group and suppose that the ranks of $\gamma_n(G)/\gamma_{n+1}(G)$ grow exponentially. Then G has uniformly exponential growth and the estimate

$$\lambda_*(G) \ge \limsup_{n \longrightarrow \infty} \sqrt[n]{\operatorname{rank}(\gamma_n(G)/\gamma_{n+1}(G))}$$

holds.

Recall that a group G is parafree of para-rank m if it is residually nilpotent and the factors of consecutive groups in its lower central series equal the corresponding ones of a free group of rank m. There are parafree groups which are not isomorphic to free groups [B 2,3].

3.3. PROPOSITION. A finitely generated parafree group G of para-rank $m \ge 2$ has uniformly exponential growth and

$$\lambda_*(G) \geq m$$
.

Proof. It is known (see for instance [MKS: Thms. 5.11 (Witt's Formulae) and 5.12]) that for a free group \mathbf{F}_m the rank of $(\gamma_n(\mathbf{F}_m)/\gamma_{n+1}(\mathbf{F}_m))$ equals the n-th coefficient of the Maclaurin power series of the function U(z) = 1/(1-mz) and the previous corollary can be applied.

Given a parafree group G of para-rank $m \ge 2$ it would be interesting to compare $\lambda_*(G)$ with $\lambda_*(\mathbf{F}_m) = 2m - 1$.

3.4. PROBLEM. Is it true that, for a finitely generated para-free group G of para-rank $m \geq 2$ which is not free, one has $\lambda_*(G) > 2m-1$?

In order to formulate the next statement we recall the following

3.5. DEFINITION. An element $R \in F$ is said to be *primitive with respect* to the lower central series if, for all $n \geq 2$, it is not an n-th power modulo $\gamma_{\omega(R)+1}(F)$ where $\omega(R)$ is the weight of R. (The latter is defined by $R \in \gamma_{\omega(R)}(F)$ but $R \notin \gamma_{\omega(R)+1}(F)$.)

3.6. THEOREM ([L 1,2]). Let R be an element of the free group F of finite rank m which is primitive with respect to the lower central series. Denote by $k = \omega(R)$ its weight and by $\langle R \rangle$ the normal closure of R in F. Let $G = F/\langle R \rangle$ and let $\mathcal{L}(F)$ and $\mathcal{L}(G)$ be the corresponding Lie algebras. Let then r be the image of R in $\mathcal{L}_k(F)$, the k-th component of $\mathcal{L}(F)$ and denote by I the ideal of $\mathcal{L}(F)$ generated by r.

Then I is the kernel of the canonical homomorphism of $\mathcal{L}(F)$ onto $\mathcal{L}(G)$, i.e.

$$\mathcal{L}(G) = \mathcal{L}(F)/I.$$

Moreover for all $n \geq 1$ the abelian group $\mathcal{L}_n(G)$ is a torsion free group whose rank is the n-th coefficient of the Maclaurin power series of the function

$$U(z) = \frac{1}{1 - mz + z^k} \cdot$$

4. More on uniformly exponential growth of one-relator groups

Any two-generated one-relator group G can be presented in the form $G = \langle a, b : a^k w(a, b) = 1 \rangle$ where $k \in \mathbb{Z}$ and w(a, b) belongs to the commutator subgroup [F, F] of the free group F = F(a, b) freely generated by a and b (this follows from Lemma 1.1). Since a and b constitute a basis in $F/\gamma_2(F)$ and [a, b] generates $\gamma_2(F)/\gamma_3(F)$, one can also present G in the form

$$G = \langle a, b : a^k[a, b]^l w(a, b) = 1 \rangle$$

where $k, l \in \mathbf{Z}$ and $w(a, b) \in \gamma_3(F)$.

In this section we shall see that, under suitable assumptions on k.l and w(a,b), the corresponding group has uniformly exponential growth.

As an application of Labute's Theorem we get the following:

4.1. PROPOSITION. Let $G = \langle a, b : R(a, b) = 1 \rangle$ be such that R is primitive with respect to $\{\gamma_n(F)\}_{n=1}^{\infty}$ and $R \in \gamma_3(F)$. Then G has uniformly exponential growth.

Proof. If $\omega(R) \geq 3$, Theorem 3.6 shows that the corresponding function U(z) has a pole z_0 with $0 < z_0 < 1$. It follows that the coefficients $c_n(G)$ grow exponentially. By Corollary 3.2, $\lambda_*(G) > 1$.