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**Autor:** Ceccherini-Silberstein, Tullio G. / GRIGORCHUK, Rostislav I.  
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Denoting now by

$$\lambda_*(G) = \inf_A \lambda_A(G)$$

the *minimal growth rate* of  $G$ , where the infimum is taken over all finite generating systems, the group  $G$  has *uniform exponential growth* if  $\lambda_*(G) > 1$ . This last concept is due to Avez [A] where the number

$$h(G) = \log(\lambda_*(G))$$

is called the *entropy* of the group  $G$  and it is discussed in [GrLP], [SW] and in the survey paper [GH].

The simplest example of a group with uniformly exponential growth is the free group  $\mathbf{F}_m$  of finite rank  $m \geq 2$  for which the minimal growth rate is  $\lambda_*(\mathbf{F}_m) = 2m - 1$ , see for instance [GH].

It is not known whether a group of exponential growth has necessarily uniformly exponential growth or not. We formulate the following :

0.1. CONJECTURE. *All one-relator groups of exponential growth have uniformly exponential growth.*

Conjecture 0.1 is true for one-relator groups of rank  $m \geq 3$  and for one-relator groups with torsion, therefore we focus our attention on two-generated one-relator groups and give sufficient conditions for such groups to have uniformly exponential growth. We present a new method for estimating the minimal growth rate of a finitely generated group using growth functions of the corresponding graded Lie algebra and apply it to one-relator groups.

## 1. AN ALGORITHM FOR CHECKING AMENABILITY

Let  $G$  be a one-relator group with presentation  $(*)$ ; the number  $m$  of the generators of  $G$  in the presentation is called the rank of the presentation. Until Section 4 we shall assume that  $R$  is cyclically reduced and non trivial.

The next observation is well known. We shall include the proof stressing the algorithmic aspect of the statement.

1.1. LEMMA. *Let  $G = \langle a, b, \dots : R(a, b, \dots) \rangle$  be a one-relator group with at least two generators. Then  $G$  has a presentation  $\langle t, \dots : R'(t, \dots) \rangle$  with  $\sigma_t(R') = 0$ , where  $\sigma_t(R')$  denotes the sum of the exponents of  $t$  in the word  $R'$ . This second presentation can in fact be produced, starting from the original one, in an algorithmical way.*

*Proof.* Let  $a$  and  $b$  be two generators involved in  $R$ ; if  $\sigma_a(R) = 0$  or  $\sigma_b(R) = 0$  we are already done. If not, suppose that  $0 < |\sigma_a(R)| \leq |\sigma_b(R)|$ ; by exchanging  $a$  with  $a^{-1}$  and/or  $b$  with  $b^{-1}$  if necessary, we can suppose that  $0 < \sigma_a(R) \leq \sigma_b(R)$ . Set  $a' = ab$  and  $b' = b$ ; then, if  $R'(a', b')$  is the expression of  $R$  in terms of the new generators  $a'$  and  $b'$ , one has  $\sigma_{a'}(R') = \sigma_a(R)$  and  $|\sigma_{b'}(R')| < \sigma_b(R)$ . Applying this procedure inductively for at most  $|\sigma_a(R)| + |\sigma_b(R)|$  times one gets the claimed presentation.  $\square$

Note that the rank of the second presentation in the previous lemma coincides with the rank of the initial one.

1.2. THEOREM. *The following is an algorithm which establishes if a given one-relator group  $G$  with presentation  $(*)$  is amenable or not:*

*Step 1: If  $m \geq 3$  then  $G$  is not amenable. If  $m = 1$  then  $G$  is amenable; if  $m = 2$  go to next step.*

*Step 2: Check if  $R$  is a power of one of the generators. If this is the case and the power is proper then  $G$  is not amenable, if  $R$  coincides, up to inversion, with one of the generators then  $G$  is amenable. Otherwise go to next step.*

*Step 3: Using the algorithm from the above lemma, change the presentation of  $G$  so that the sum of the exponents of one of the generators in the relator is zero. Then  $G$  is amenable iff, up to a relabeling and inversion of the generators, and up to a cyclic permutation of the relator, the presentation obtained is of the form  $\langle t, s : tst^{-1}s^{-n} = 1 \rangle$ , with  $n \in \mathbf{Z} \setminus \{0\}$ .*

*Proof.* Recall that the Freiheitssatz of Wilhelm Magnus ([MKS: Thm. 4.10] and [LS: IV Thm. 5.1]) states that, if  $R = R(a_1, a_2, \dots, a_m)$  is a cyclically reduced word in  $a_1, a_2, \dots, a_m$  and involves  $a_m$ , then the subgroup of  $G = \langle a_1, a_2, \dots, a_m : R(a_1, a_2, \dots, a_m) = 1 \rangle$  generated by  $a_1, a_2, \dots, a_{m-1}$  is freely generated by them.

(1) If  $m \geq 3$  then, by Magnus' Theorem,  $G$  contains the free group on two generators and thus it is not amenable. If  $m = 1$  then  $G = \langle a : a^n = 1 \rangle$  is cyclic and therefore amenable.

(2) Let  $m = 2$ . If  $R$  is a proper power of one of the generators, say  $R = a^n$  with  $|n| \geq 2$ , then  $G$  is isomorphic to the free product  $\mathbf{Z} * \mathbf{Z}_{|n|}$  of the infinite cyclic group and the cyclic group of order  $|n| \geq 2$  and it is not amenable because its commutator subgroup is a free group of infinite rank. If  $R$  coincides, up to inversion, with one of the generators then  $G$  is infinite cyclic and therefore amenable.

(3) Suppose now that  $\langle a, b : R(a, b) = 1 \rangle$  is a presentation of  $G$  with  $\sigma_a(R) = 0$ . If we denote by  $b_i = a^i b a^{-i}$ ,  $i \in \mathbf{Z}$ , then the relator  $R$  can be expressed as a word in the  $b_i$ 's just replacing each  $b^k$  in  $R(a, b)$  by  $b_j^k$ , where  $j$  is the sum of the exponents of  $a$  in the subword of  $R$  preceding the given occurrence of  $b^k$ . We shall denote this word by  $R'(b_m, b_{m+1}, \dots, b_M)$ , where  $m$  and  $M$  are the minimum and, respectively, the maximum subscript occurring in the expression of  $R'$ . Note that since  $R(a, b)$  is cyclically reduced, then  $R'$  is cyclically reduced as well and  $m < M$ .

It is known [LS: IV, proof of Thm. 5.1] that any one-relator group with  $\geq 2$  generators is an HNN-extension  $(H; A, B, \phi)$  of another one-relator group  $H$ . In our situation

$$\begin{aligned} H &= \langle b_m, b_{m+1}, \dots, b_M; R'(b_m, b_{m+1}, \dots, b_M) \rangle \\ A &= \text{subgroup of } H \text{ generated by } b_m, b_{m+1}, \dots, b_{M-1} \\ B &= \text{subgroup of } H \text{ generated by } b_{m+1}, b_{m+2}, \dots, b_M \\ \phi : A \ni b_i &\longmapsto b_{i+1} \in B, \quad i = m, m+1, \dots, M-1. \end{aligned}$$

Therefore  $G$  also admits the following presentation

$$G = \langle a, b_m, \dots, b_M : R'(b_m, \dots, b_M) = 1, ab_i a^{-1} = b_{i+1}, i = m, \dots, M-1 \rangle.$$

The subgroups  $A$  and  $B$  are free of rank  $M-m$  and if  $M-m \geq 2$  then  $G$  is not amenable.

Suppose now that  $M-m=1$ , so that  $A = \langle b_m \rangle \cong B = \langle b_M \rangle \cong \mathbf{Z}$ . It is known ([H: Prop. 3.3]) that an HNN-extension  $(H; A, B, \phi)$ , such that  $A$  and  $B$  are both proper subgroups of the base group  $H$ , contains the free group  $\mathbf{F}_2$ . Thus, if  $A \neq H \neq B$ , then  $G$  is non amenable.

Suppose that  $A = H$  (the case  $B = H$  is similar). Then  $H = \langle b_m \rangle \cong \mathbf{Z}$  and  $b_M = b_m^k$  for a suitable  $k \in \mathbf{Z} \setminus \{0\}$ . Replacing  $a$  by  $t$  and  $b_m$  by  $s$  in the above presentation for  $G$ , one gets the presentation

$$G = \langle t, s : tst^{-1} = s^k \rangle$$

of type 3<sub>b</sub>. from the list (\*\*) and so  $G$  is amenable.  $\square$

1.3. COROLLARY. *For amenable one-relator groups the isomorphism problem is solvable.*

*Proof.* Suppose two one-relator groups which are amenable are given. Then, in the algorithmical way described above, one gets two presentations from the list (\*\*) and the procedure of recognition becomes obvious since any two groups from the list with different presentations are in fact non-isomorphic.  $\square$