

Zeitschrift: L'Enseignement Mathématique
Herausgeber: Commission Internationale de l'Enseignement Mathématique
Band: 43 (1997)
Heft: 3-4: L'ENSEIGNEMENT MATHÉMATIQUE

Artikel: AMENABILITY AND GROWTH OF ONE-RELATOR GROUPS
Autor: Ceccherini-Silberstein, Tullio G. / GRIGORCHUK, Rostislav I.
Kapitel: 0. Introduction
DOI: <https://doi.org/10.5169/seals-63284>

Nutzungsbedingungen

Die ETH-Bibliothek ist die Anbieterin der digitalisierten Zeitschriften auf E-Periodica. Sie besitzt keine Urheberrechte an den Zeitschriften und ist nicht verantwortlich für deren Inhalte. Die Rechte liegen in der Regel bei den Herausgebern beziehungsweise den externen Rechteinhabern. Das Veröffentlichen von Bildern in Print- und Online-Publikationen sowie auf Social Media-Kanälen oder Webseiten ist nur mit vorheriger Genehmigung der Rechteinhaber erlaubt. [Mehr erfahren](#)

Conditions d'utilisation

L'ETH Library est le fournisseur des revues numérisées. Elle ne détient aucun droit d'auteur sur les revues et n'est pas responsable de leur contenu. En règle générale, les droits sont détenus par les éditeurs ou les détenteurs de droits externes. La reproduction d'images dans des publications imprimées ou en ligne ainsi que sur des canaux de médias sociaux ou des sites web n'est autorisée qu'avec l'accord préalable des détenteurs des droits. [En savoir plus](#)

Terms of use

The ETH Library is the provider of the digitised journals. It does not own any copyrights to the journals and is not responsible for their content. The rights usually lie with the publishers or the external rights holders. Publishing images in print and online publications, as well as on social media channels or websites, is only permitted with the prior consent of the rights holders. [Find out more](#)

Download PDF: 12.01.2026

ETH-Bibliothek Zürich, E-Periodica, <https://www.e-periodica.ch>

AMENABILITY AND GROWTH OF ONE-RELATOR GROUPS

by Tullio G. CECCHERINI-SILBERSTEIN and Rostislav I. GRIGORCHUK

ABSTRACT. An algorithm showing whether a group given by a one-relator presentation is amenable or not is constructed. Sufficient conditions for a one-relator group of exponential growth to have uniformly exponential growth are also given.

0. INTRODUCTION

A one-relator group is a group G which admits a presentation

$$(*) \quad G = \langle a_1, a_2, \dots, a_m : R(a_1, a_2, \dots, a_m) = 1 \rangle$$

with one defining relation.

The paper by G. Baumslag [B 1] is a comprehensive survey of results about one-relator groups. In particular this paper stresses the role of algorithmic problems in the theory of one-relator groups.

Recently the interest in functional-analytical and asymptotical properties of one-relator groups has increased. For instance, the entropy of one-relator groups was discussed in [GrLP], random walks and Markov operators on one-relator groups were investigated in [CV], [BCCH], [BC], and the K-functor of reduced C^* -algebras of one-relator groups was computed in [BBV]. Also the growth functions of the groups $\Gamma_n = \langle t, a : tat^{-1} = a^n \rangle$, $n \neq 0, \pm 1$, and of some other one-relator groups were calculated in [CEG] and [EJ].

Recall that a discrete group G is amenable if there exists a finitely additive measure $\mu: \mathcal{P}(G) = \{0, 1\}^G \rightarrow [0, 1]$ which is G -(left)-invariant ($\mu(gE) = \mu(E)$ for all $g \in G$ and $E \subset G$) and such that, in addition, $\mu(G) = 1$. For our purpose it will be enough to know that a group containing a free subgroup of rank two is not amenable, and that, on the contrary, any solvable group is amenable ([G]).

As easily follows from the paper of Karrass and Solitar [KS], all amenable one-relator groups are in the following list:

$\left. \begin{array}{l} 1. \langle a : a^n = 1 \rangle \cong \mathbf{Z}_n, \text{ cyclic groups of finite order } n = 1, 2, \dots; \\ 2. \langle a, b : b = 1 \rangle \cong \mathbf{Z}, \text{ the infinite cyclic group;} \\ 3. \langle a, b : bab^{-1} = a^n \rangle, n \neq 0. \end{array} \right\} \text{This class splits into two subclasses:}$

$3_a. n = +1 : \langle a, b : bab^{-1} = a \rangle \cong \mathbf{Z}^2;$
 $n = -1 : \langle a, b : bab^{-1} = a^{-1} \rangle :$
 this group contains a subgroup $\cong \mathbf{Z}^2$ of index two,
 but it is not $\cong \mathbf{Z}^2$;
 $3_b. n \neq 0, \pm 1 : \langle a, b : bab^{-1} = a^n \rangle :$
 these groups are 2 step-solvable and of exponential
 growth (pairwise non-isomorphic).

Also Tits' alternative does hold for one-relator groups: any one-relator group either contains a free subgroup of rank two or is solvable (and from the above list).

But in the Karrass-Solitar paper no algorithm is given answering the question whether, given a one-relator presentation, the corresponding group is solvable or not. In Section 1 we present a simple algorithm and, as a consequence, we re-obtain the above list of all amenable one-relator groups.

In the second part of the paper we investigate the uniformly exponential growth for one-relator groups of exponential growth.

Recall that if G is a group with a finite generating system A ,

$$|g|_A = \min \{n : g = a_1 a_2 \cdots a_n, a_i \in A\}$$

is the *length* of an element $g \in G$ with respect to A and $\gamma_A^G(n) = |\{g \in G : |g|_A \leq n\}|$ is the *growth function* of G with respect to the generating system A . The limit

$$\lambda_A(G) = \lim_{n \rightarrow \infty} \sqrt[n]{\gamma_A^G(n)}$$

exists and $\lambda_A(G) \geq 1$. The group G is said to have *exponential growth* (respectively *sub-exponential growth*) if $\lambda_A(G) > 1$ (resp. $\lambda_A(G) = 1$) for some (and therefore for any other) finite system of generators A .

Denoting now by

$$\lambda_*(G) = \inf_A \lambda_A(G)$$

the *minimal growth rate* of G , where the infimum is taken over all finite generating systems, the group G has *uniform exponential growth* if $\lambda_*(G) > 1$. This last concept is due to Avez [A] where the number

$$h(G) = \log(\lambda_*(G))$$

is called the *entropy* of the group G and it is discussed in [GrLP], [SW] and in the survey paper [GH].

The simplest example of a group with uniformly exponential growth is the free group \mathbf{F}_m of finite rank $m \geq 2$ for which the minimal growth rate is $\lambda_*(\mathbf{F}_m) = 2m - 1$, see for instance [GH].

It is not known whether a group of exponential growth has necessarily uniformly exponential growth or not. We formulate the following :

0.1. CONJECTURE. *All one-relator groups of exponential growth have uniformly exponential growth.*

Conjecture 0.1 is true for one-relator groups of rank $m \geq 3$ and for one-relator groups with torsion, therefore we focus our attention on two-generated one-relator groups and give sufficient conditions for such groups to have uniformly exponential growth. We present a new method for estimating the minimal growth rate of a finitely generated group using growth functions of the corresponding graded Lie algebra and apply it to one-relator groups.

1. AN ALGORITHM FOR CHECKING AMENABILITY

Let G be a one-relator group with presentation $(*)$; the number m of the generators of G in the presentation is called the rank of the presentation. Until Section 4 we shall assume that R is cyclically reduced and non trivial.

The next observation is well known. We shall include the proof stressing the algorithmic aspect of the statement.

1.1. LEMMA. *Let $G = \langle a, b, \dots : R(a, b, \dots) \rangle$ be a one-relator group with at least two generators. Then G has a presentation $\langle t, \dots : R'(t, \dots) \rangle$ with $\sigma_t(R') = 0$, where $\sigma_t(R')$ denotes the sum of the exponents of t in the word R' . This second presentation can in fact be produced, starting from the original one, in an algorithmical way.*