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AMENABILITY AND GROWTH OF ONE-RELATOR GROUPS

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ABSTRACT. An algorithm showing whether a group given by a one-relator presentation is amenable or not is constructed. Sufficient conditions for a one-relator group of exponential growth to have uniformly exponential growth are also given.

0. INTRODUCTION

A one-relator group is a group G which admits a presentation

$$(*) \qquad G = \langle a_1, a_2, \dots, a_m \ : \ R(a_1, a_2, \dots, a_m) = 1 \rangle$$

with one defining relation.

The paper by G. Baumslag [B 1] is a comprehensive survey of results about one-relator groups. In particular this paper stresses the role of algorithmic problems in the theory of one-relator groups.

Recently the interest in functional-analytical and asymptotical properties of one-relator groups has increased. For instance, the entropy of one-relator groups was discussed in [GrLP], random walks and Markov operators on one-relator groups where investigated in [CV], [BCCH], [BC], and the K-functor of reduced C^*-algebras of one-relator groups was computed in [BBV]. Also the growth functions of the groups $\Gamma_n = \langle t, a : tat^{-1} = a^n \rangle$, $n \neq 0, \pm 1$, and of some other one-relator groups were calculated in [CEG] and [EJ].

Recall that a discrete group G is amenable if there exists a finitely additive measure $\mu \colon \mathcal{P}(G) = \{0,1\}^G \longrightarrow [0,1]$ which is G-(left)-invariant ($\mu(gE) = \mu(E)$ for all $g \in G$ and $E \subset G$) and such that, in addition, $\mu(G) = 1$. For our purpose it will be enough to know that a group containing a free subgroup of rank two is not amenable, and that, on the contrary, any solvable group is amenable ([G]).

As easily follows from the paper of Karrass and Solitar [KS], all amenable one-relator groups are in the following list:

(**)

1. $\langle a : a^n = 1 \rangle \cong \mathbf{Z}_n$, cyclic groups of finite order $n = 1, 2, \ldots$;
2. $\langle a, b : b = 1 \rangle \cong \mathbf{Z}$, the infinite cyclic group;
3. $\langle a, b : bab^{-1} = a^n \rangle$, $n \neq 0$.

   This class splits into two subclasses:

   3_a. $n = +1$: $\langle a, b : bab^{-1} = a \rangle \cong \mathbf{Z}^2$;

   $n = -1$: $\langle a, b : bab^{-1} = a^{-1} \rangle$:

   this group contains a subgroup $\cong \mathbf{Z}^2$ of index two, but it is not $\cong \mathbf{Z}^2$;

   3_b. $n \neq 0, \pm 1$: $\langle a, b : bab^{-1} = a^n \rangle$:

   these groups are 2 step-solvable and of exponential growth (pairwise non-isomorphic).

Also Tits' alternative does hold for one-relator groups: any one-relator group either contains a free subgroup of rank two or is solvable (and from the above list).

But in the Karrass-Solitar paper no algorithm is given answering the question whether, given a one-relator presentation, the corresponding group is solvable or not. In Section 1 we present a simple algorithm and, as a consequence, we re-obtain the above list of all amenable one-relator groups.

In the second part of the paper we investigate the uniformly exponential growth for one-relator groups of exponential growth.

Recall that if G is a group with a finite generating system A,

$$|g|_A = \min\{n : g = a_1 a_2 \cdots a_n,\ a_i \in A\}$$

is the *length* of an element $g \in G$ with respect to A and $\gamma_A^G(n) = |\{g \in G : |g|_A \leq n\}|$ is the *growth function* of G with respect to the generating system A. The limit

$$\lambda_A(G) = \lim_{n \longrightarrow \infty} \sqrt[n]{\gamma_A^G(n)}$$

exists and $\lambda_A(G) \geq 1$. The group G is said to have *exponential growth* (respectively *sub-exponential growth*) if $\lambda_A(G) > 1$ (resp. $\lambda_A(G) = 1$) for some (and therefore for any other) finite system of generators A.

Denoting now by

$$\lambda_*(G) = \inf_A \lambda_A(G)$$

the *minimal growth rate* of G, where the infimum is taken over all finite generating systems, the group G has *uniform exponential growth* if $\lambda_*(G) > 1$. This last concept is due to Avez [A] where the number

$$h(G) = \log(\lambda_*(G))$$

is called the *entropy* of the group G and it is discussed in [GrLP], [SW] and in the survey paper [GH].

The simplest example of a group with uniformly exponential growth is the free group $\mathbf{F}_m$ of finite rank $m \geq 2$ for which the minimal growth rate is $\lambda_*(\mathbf{F}_m) = 2m - 1$, see for instance [GH].

It is not known whether a group of exponential growth has necessarily uniformly exponential growth or not. We formulate the following:

0.1. CONJECTURE. *All one-relator groups of exponential growth have uniformly exponential growth.*

Conjecture 0.1 is true for one-relator groups of rank $m \geq 3$ and for one-relator groups with torsion, therefore we focus our attention on two-generated one-relator groups and give sufficient conditions for such groups to have uniformly exponential growth. We present a new method for estimating the minimal growth rate of a finitely generated group using growth functions of the corresponding graded Lie algebra and apply it to one-relator groups.

1. AN ALGORITHM FOR CHECKING AMENABILITY

Let G be a one-relator group with presentation $(*)$; the number m of the generators of G in the presentation is called the rank of the presentation. Untill Section 4 we shall assume that R is cyclically reduced and non trivial.

The next observation is well known. We shall include the proof stressing the algorithmic aspect of the statement.

1.1. LEMMA. *Let $G = \langle a, b, \dots : R(a, b, \dots) \rangle$ be a one-relator group with at least two generators. Then G has a presentation $\langle t, \dots : R'(t, \dots) \rangle$ with $\sigma_t(R') = 0$, where $\sigma_t(R')$ denotes the sum of the exponents of t in the word R'. This second presentation can in fact be produced, starting from the original one, in an algorithmical way.*