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map $H^1(G; \mathbf{Z}G) \longrightarrow H^1(G; l_2G)$ induced by the imbedding $\mathbf{Z}G \longrightarrow l_2G$ is easily seen to be injective. Since we have assumed $H^1(G; \mathbf{Z}G) \neq 0$ the result follows.

5. The vanishing of $q(G)$

5.1. Here we mention in a few words what happens when for a finitely presented group G the invariant $q(G)$ is 0. For the details and more comments we refer to the paper [E2]. We thus consider a 4-manifold M with $\pi_1(M) = G$ and $\chi(M) = 0$.

Since we restrict attention to groups with $\overline{\beta}_1(G) = 0$ the vanishing of $\chi(M)$ implies $\overline{\beta}_2(M) = 0$, whence $\overline{H}^2(\widetilde{M}) = 0$. As shown in [E2] by a spectral sequence argument it follows that $H^2(M; \mathbf{Z}G)$ is isomorphic to $H^2(G; \mathbf{Z}G)$, ordinary cohomology with local coefficients $\mathbf{Z}G$. By Poincaré duality $H^2(M; \mathbf{Z}G) = H_2(M; \mathbf{Z}G)$ which can be identified with $H_2(\widetilde{M}; \mathbf{Z})$. Since $\widetilde{M}$ is simply connected, $H_2(\widetilde{M}; \mathbf{Z})$ is isomorphic to the second homotopy group $\pi_2(\widetilde{M}) = \pi_2(M)$.

What about $H_3(\widetilde{M}; \mathbf{Z})$? It can be identified with $H_3(M; \mathbf{Z}G)$ which, by Poincaré duality, is $\cong H^1(M; \mathbf{Z}G) = H^1(G; \mathbf{Z}G)$. This group, the "endpoint-group" of G, is known to be either 0 or $\mathbf{Z}$ or of infinite rank. As mentioned in 4.4, remark 3) the latter case is excluded by our assumption $\overline{\beta}_1(G) = 0$. The case $H^1(G; \mathbf{Z}G) = \mathbf{Z}$ is exceptional: it means that G is virtually infinite cyclic, and we exclude this. Then $H_3(\widetilde{M}; \mathbf{Z}) = 0$.

5.2. We now add the assumption that $H^2(G; \mathbf{Z}G) = 0$. This is a property shared by many groups (e.g. duality groups). Then the homology groups $H_i(\widetilde{M}; \mathbf{Z})$ are $= 0$ for $i = 1, 2, 3, 4$ ($i = 4$ because $\widetilde{M}$ is an open manifold). Thus all homotopy groups of $\widetilde{M}$ are $= 0$, $\widetilde{M}$ is contractible, M is a $K(G, 1)$, and the group G fulfills Poincaré duality.

THEOREM 6. *Let G be an infinite, finitely presented group, not virtually infinite cyclic, fulfilling $\overline{\beta}_1(G) = 0$ and $H^2(G; \mathbf{Z}G) = 0$, and let M be a manifold with fundamental group G. If the Euler characteristic $\chi(M) = 0$, then M is an Eilenberg-MacLane space for G and G is a Poincaré duality group of dimension* 4.

We recall that for knot groups and 2-knot groups $q(G) = 0$, see examples 3) and 4) in 2.2. Theorem 6 can only be applied to 2-knot groups which are not classical knot groups since the latter have cohomological dimension 2.