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map  $H^1(G; \mathbf{Z}G) \longrightarrow H^1(G; l_2 G)$  induced by the imbedding  $\mathbf{Z}G \longrightarrow l_2 G$  is easily seen to be injective. Since we have assumed  $H^1(G; \mathbf{Z}G) \neq 0$  the result follows.

## 5. THE VANISHING OF $q(G)$

5.1. Here we mention in a few words what happens when for a finitely presented group  $G$  the invariant  $q(G)$  is 0. For the details and more comments we refer to the paper [E2]. We thus consider a 4-manifold  $M$  with  $\pi_1(M) = G$  and  $\chi(M) = 0$ .

Since we restrict attention to groups with  $\bar{\beta}_1(G) = 0$  the vanishing of  $\chi(M)$  implies  $\bar{\beta}_2(M) = 0$ , whence  $\bar{H}^2(\tilde{M}) = 0$ . As shown in [E2] by a spectral sequence argument it follows that  $H^2(M; \mathbf{Z}G)$  is isomorphic to  $H^2(G; \mathbf{Z}G)$ , ordinary cohomology with local coefficients  $\mathbf{Z}G$ . By Poincaré duality  $H^2(M; \mathbf{Z}G) = H_2(M; \mathbf{Z}G)$  which can be identified with  $H_2(\tilde{M}; \mathbf{Z})$ . Since  $\tilde{M}$  is simply connected,  $H_2(\tilde{M}; \mathbf{Z})$  is isomorphic to the second homotopy group  $\pi_2(\tilde{M}) = \pi_2(M)$ .

What about  $H_3(\tilde{M}; \mathbf{Z})$ ? It can be identified with  $H_3(M; \mathbf{Z}G)$  which, by Poincaré duality, is  $\cong H^1(M; \mathbf{Z}G) = H^1(G; \mathbf{Z}G)$ . This group, the “endpoint-group” of  $G$ , is known to be either 0 or  $\mathbf{Z}$  or of infinite rank. As mentioned in 4.4, remark 3) the latter case is excluded by our assumption  $\bar{\beta}_1(G) = 0$ . The case  $H^1(G; \mathbf{Z}G) = \mathbf{Z}$  is exceptional: it means that  $G$  is virtually infinite cyclic, and we exclude this. Then  $H_3(\tilde{M}; \mathbf{Z}) = 0$ .

5.2. We now add the assumption that  $H^2(G; \mathbf{Z}G) = 0$ . This is a property shared by many groups (e.g. duality groups). Then the homology groups  $H_i(\tilde{M}; \mathbf{Z})$  are = 0 for  $i = 1, 2, 3, 4$  ( $i = 4$  because  $\tilde{M}$  is an open manifold). Thus all homotopy groups of  $\tilde{M}$  are = 0,  $\tilde{M}$  is contractible,  $M$  is a  $K(G, 1)$ , and the group  $G$  fulfills Poincaré duality.

**THEOREM 6.** *Let  $G$  be an infinite, finitely presented group, not virtually infinite cyclic, fulfilling  $\bar{\beta}_1(G) = 0$  and  $H^2(G; \mathbf{Z}G) = 0$ , and let  $M$  be a manifold with fundamental group  $G$ . If the Euler characteristic  $\chi(M) = 0$ , then  $M$  is an Eilenberg-MacLane space for  $G$  and  $G$  is a Poincaré duality group of dimension 4.*

We recall that for knot groups and 2-knot groups  $q(G) = 0$ , see examples 3) and 4) in 2.2. Theorem 6 can only be applied to 2-knot groups which are not classical knot groups since the latter have cohomological dimension 2.