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Autor: TABACHNIKOV, Serge

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REMARK. The following result is also known (see the literature cited): if a convex closed curve intersects a curve, homothetic to J, at 2n points then it has at least 2n Minkowski vertices.

# 5. Conservative transverse line fields

In this section we discuss the following problem: given a smooth strictly convex closed plane curve  $\gamma$  and a smooth transverse line field l along it, when does a parameterization  $\gamma(t)$  exist such that the line l(t) at point  $\gamma(t)$  is generated by the acceleration vector  $\gamma''(t)$  for all t?

DEFINITION. A transverse line field along a closed plane curve, generated by the acceleration vectors for some parameterization of the curve, is called *conservative*.

Clearly, not every line field is conservative: consider, for example, a field of lines that everywhere make an acute angle with the curve. Theorem 0.1 provides a necessary condition: the envelope of the lines from a conservative line field has at least 4 cusps. Lemma 3.2 gives another one: there exist at least 2 tangent lines to this envelope through every point in the plane.

We start with the following situation. Let  $M^3$  be a contact manifold and let  $\widetilde{\gamma} \subset M$  be a closed smooth Legendrian curve. Recall that the characteristic line field  $\eta$  of a contact form  $\lambda$  is the field  $Ker\ d\lambda$ . Assume that the contact distribution along  $\widetilde{\gamma}$  is coorientable; then it can be determined by a contact form. Let  $\eta$  be a line field along  $\widetilde{\gamma}$ , transverse to the contact distribution.

QUESTION. When does a contact form exist in a vicinity of  $\widetilde{\gamma}$  for which  $\eta$  is the characteristic field?

When this is the case we call the field  $\eta$  characteristic.

Let  $\lambda$  be some contact form near  $\widetilde{\gamma}$  and let v be a vector field along  $\widetilde{\gamma}$  that generates the line field  $\eta$ . Consider the 1-form  $(i_v \ d\lambda)/\lambda(v)$  and set

$$\beta(\widetilde{\gamma},\eta) = \int_{\widetilde{\gamma}} \frac{i_v \ d\lambda}{\lambda(v)} \cdot$$

THEOREM 5.1. The number  $\beta(\tilde{\gamma}, \eta)$  does not depend on the choice of the contact form  $\lambda$  nor the vector field v. This number vanishes if and only if the field  $\eta$  is characteristic.

*Proof.* Clearly,  $(i_v \ d\lambda)/\lambda(v)$  does not change if v is multiplied by a nonvanishing function. Let  $\lambda_1 = f\lambda$  with  $f \neq 0$  be another contact form. Then  $d\lambda_1 = df \wedge \lambda + fd\lambda$ . One has

$$\int_{\widetilde{\gamma}} \frac{i_v \ d\lambda_1}{\lambda_1(v)} = \int_{\widetilde{\gamma}} \frac{f \ i_v \ d\lambda + df(v) \ \lambda - \lambda(v) \ df}{f \ \lambda(v)}$$
$$= \int_{\widetilde{\gamma}} \frac{i_v \ d\lambda}{\lambda(v)} + \int_{\widetilde{\gamma}} \frac{df(v)}{f \ \lambda(v)} \lambda - \int_{\widetilde{\gamma}} \frac{df}{f} \cdot \frac{df(v)}{f \ \lambda(v)} \lambda - \int_{\widetilde{\gamma}} \frac{df}{f} \cdot \frac{df}{f} \cdot$$

The second integral on the right hand side vanishes because  $\widetilde{\gamma}$  is a Legendrian curve, tangent to the kernel of  $df(v)\lambda/f\lambda(v)$ , and so does the third because df/f is an exact 1-form. Thus  $\beta(\widetilde{\gamma}, \eta)$  does not depend on the choices involved.

If  $\eta$  is characteristic for a contact form  $\lambda$  then  $i_v d\lambda = 0$ , so  $\beta(\widetilde{\gamma}, \eta) = 0$ . Conversely, let  $\beta(\widetilde{\gamma}, \eta) = 0$ . A neighbourhood of  $\widetilde{\gamma}$  in M is contactomorphic to a neighbourhood of the zero section in the space of 1-jets  $J^1S^1$  (see [A 3]). That is, there exist coordinates (x, y, z),  $x \in S^1$ ,  $y, z \in \mathbf{R}^1$  in which the contact structure is given by the 1-form  $\lambda_0 = dz - ydx$ , and  $\widetilde{\gamma}$  is the curve y = z = 0. Since  $\eta$  is transverse to the contact structure one may assume it to be generated by the vector field

$$v = a(x) \partial/\partial x + b(x) \partial/\partial y + \partial/\partial z$$
,

where a(x) and b(x) are functions on the circle.

Then

$$\beta(\widetilde{\gamma}, \eta) = \int_{\widetilde{\gamma}} \frac{i_v \ d\lambda_0}{\lambda_0(v)} = -\int b(x) \ dx.$$

If  $\beta(\widetilde{\gamma}, \eta)$  vanishes then there exists a function g(x) such that b(x) = g'(x). Next, a direct computation shows that the characteristic line field of the contact form  $e^{f(x,y,z)}$   $\lambda_0$  is generated by the vector field

$$f_y \partial/\partial x - (f_x + yf_z) \partial/\partial y + (1 + yf_y) \partial/\partial z$$
,

which equals, along  $\widetilde{\gamma}$ ,

$$u = f_y \ \partial/\partial x - f_x \ \partial/\partial y + \partial/\partial z.$$

Therefore, setting f(x, y, z) = a(x)y - g(x), one has: v = u, and the field  $\eta$  is characteristic.

Thus the characteristic line fields constitute a codimension 1 subspace in the (infinite dimensional) space of line fields along  $\tilde{\gamma}$ , transverse to the contact structure.

Return to the situation at the beginning of the section. Let  $\gamma$  be a smooth strictly convex closed curve, cooriented inwards, and let l be a smooth

transverse line field along  $\gamma$ . As before,  $\widetilde{\gamma}$  is the Legendrian curve in the space of cooriented contact elements  $ST^*\mathbf{R}^2$ , corresponding to  $\gamma$ . For every point  $x \in \gamma$  consider the family of cooriented contact elements along the line l(x), parallel to the contact element of  $\gamma$  at x. This gives a line field  $\eta$  along  $\widetilde{\gamma}$ , a lift of the field l. The field  $\eta$  is transverse to the contact structure.

Choose a parameterization  $\gamma(t)$ ,  $0 \le t \le T$ , and a vector field u(t) along  $\gamma$  that generates the line field l(t).

LEMMA 5.2. One has:

$$\beta(\widetilde{\gamma},\eta) = \int_0^T \frac{\left[\gamma''(t),u(t)\right]}{\left[\gamma'(t),u(t)\right]} dt.$$

*Proof.* Let v be the lift of u to  $ST^*\mathbf{R}^2$  that generates the field  $\eta$ . In Theorem 2.1 a Hamiltonian function H in  $ST^*\mathbf{R}^2$  is constructed, associated with the parameterization  $\gamma(t)$  (one does not need the assumption  $\left[\gamma''(t), \gamma'''(t)\right] \neq 0$  here). The space  $ST^*\mathbf{R}^2$  is identified with  $\mathbf{R}^2 \times S$ , where the star-shaped curve  $S \subset (\mathbf{R}^2)^*$ , the level curve of H, consists of the covectors  $[\gamma'(t), ]$ . The corresponding contact form  $\lambda$  is the restriction of the Liouville form pdq to  $\mathbf{R}^2 \times S$ . The curve  $\widetilde{\gamma}$  is given by the formula:

$$\widetilde{\gamma}(t) = (\gamma(t), [\gamma'(t),]).$$

It follows that  $\lambda(v(t)) = [\gamma'(t), u(t)]$ . Likewise,

$$(i_{v(t)}d\lambda)(\widetilde{\gamma}'(t)) = (i_{v(t)}dp \wedge dq)(\widetilde{\gamma}'(t)) = [\gamma''(t), u(t)].$$

Therefore

$$\int_{\widetilde{\gamma}} \frac{i_v \ d\lambda}{\lambda(v)} = \int_0^T \frac{\left[\gamma''(t), u(t)\right]}{\left[\gamma'(t), u(t)\right]} dt.$$

The lemma is proved.

In particular, the value of the integral

$$\int_0^T \frac{\left[\gamma''(t), u(t)\right]}{\left[\gamma'(t), u(t)\right]} dt$$

does not depend on the parameterization  $\gamma(t)$  nor on the choice of the vector field u(t). Denote this integral by  $\alpha(\gamma, l)$ .

LEMMA 5.3. The line field l along  $\gamma$  is conservative if and only if the line field  $\eta$  along  $\widetilde{\gamma}$  is characteristic.

*Proof.* If l is generated by the vectors  $\gamma''(t)$  then  $\eta$  consists of the characteristic directions of the contact form in  $ST^*\mathbf{R}^2$ , associated with the parameterization  $\gamma(t)$  in Theorem 2.1 (cf. the proof of the preceding lemma).

Conversely, a contact form  $\lambda$  along  $\widetilde{\gamma}$ , whose characteristics are the lines  $\eta$ , is a field of covectors p along  $\gamma$  which vanish on the tangent lines to  $\gamma$  at the respective points. Define the parameterization  $\gamma(t)$  by the condition:  $[\gamma'(t), ] = p \ (\gamma(t))$  for all t. Then the contact form in  $ST^*\mathbf{R}^2$ , associated with this parameterization according to Theorem 2.1, coincides with  $\lambda$  along  $\widetilde{\gamma}$ . Therefore the lines l(t) are generated by the vectors  $\gamma''(t)$ .

Combining Theorem 5.1, Lemma 5.2 and 5.3, one arrives at the following result (discovered in [T 2] and proved therein by a direct computation).

Theorem 5.4. A transverse line field l along a smooth strictly convex closed plane curve  $\gamma$  is conservative if and only if  $\alpha(\gamma, l) = 0$ .

Thus conservative line fields constitute a codimension one subspace in the space of transverse line fields along a closed curve.

EXAMPLE. L. Guieu and V. Ovsienko studied the following situation in [G-O]. Given a smooth convex closed plane curve consider the field of lines connecting each point of the curve with a focus of its osculating conic at this point (see Example 2 in Section 3). This line field is conservative, and its envelope, called the gravitational caustic in [G-O], has at least 6 cusps.

Consider a curve  $\gamma$  with a transverse line field l. A (partial) diffeomorphism of the plane F takes  $\gamma$  to a new curve  $F(\gamma)$  with the transverse line field dF(l). The field dF(l) does not have to be conservative even if l is.

EXAMPLE. Let  $\gamma$  be the unit circle, l consists of its normals, and F is given near  $\gamma$  in polar coordinates by the formula:  $(\alpha, r) \to (\alpha + r, r)$ . Then  $F(\gamma) = \gamma$ , and the lines dF(l) make a constant acute angle with the circle.

However the following result holds (to answer a question by V. Arnold).

THEOREM 5.5. Every projective transformation of the plane takes the conservative line fields to the conservative ones.

*Proof.* Consider  $\mathbb{R}^2$  as the plane  $\{z=1\}$  in Euclidean 3-space, and let

$$\pi:(x,y,z)\to(x/z,y/z)$$

be the projection of the half-space  $\mathbf{R}_+^3 = \{z > 0\}$  on  $\mathbf{R}^2$ . Consider a parametrized curve  $\Gamma(t) \subset \mathbf{R}_+^3$ , and let  $\gamma(t) = \pi(\Gamma(t))$ .

Claim: the field  $(d\pi)(\Gamma''(t))$  is conservative along the curve  $\gamma(t)$ .

Indeed, a direct computation (which is left to the reader) shows that

$$(d\pi)\big(\Gamma''(t)\big) = \gamma''(t) + 2 \frac{z'(t)}{z(t)} \gamma'(t).$$

Therefore

$$\alpha(\gamma, (d\pi)(\Gamma''(t))) = -\int 2 \frac{z'(t)}{z(t)} dt = -2 \int d \log z(t) = 0.$$

The claim follows from Theorem 5.4.

Let A be a linear transformation of space. Then  $F = \pi A : \mathbf{R}^2 \to \mathbf{R}^2$  is a projective transformation, and all projective transformations are obtained this way. Consider a curve  $\gamma(t) \subset \mathbf{R}^2$ , and let l(t) be generated by the acceleration vectors  $\gamma''(t)$ . Let  $\Gamma(t) = A(\gamma(t))$ ; assume, without loss of generality, that  $\Gamma(t) \subset \mathbf{R}^3_+$  One has:  $\Gamma''(t) = A(\gamma''(t))$ , and it follows from the above claim that the field  $(d\pi)(\Gamma''(t))$  is conservative along the curve  $\pi(\Gamma(t))$ . Thus the line field dF(l) is conservative along the curve  $F(\gamma)$ .

REMARK. Theorem 5.5 shows that the notion of the conservative line fields along closed curves is a projective, and not an affine, one. Thus one hopes that the theory of this paper can be extended to spherical curves in the spirit of [A 5].

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ADDED IN PROOF. A higher dimensional analog of conservative transverse line fields is studied in the author's paper "Exact transverse line fields and projective billiards in a ball", to appear in "Geometric and Functional Analysis".

## **REFERENCES**

- [A 1] ARNOLD, V. *Topological Invariants of Plane Curves and Caustics*. University Lecture Series, v. 5, AMS, 1994.
- [A 2] Mathematical Methods of Classical Mechanics. Springer-Verlag, 1989.
- [A 3] Contact geometry and wave propagation. L'Enseign. Math. 36 (1990), 215–266.
- [A 4] On the number of flattening points on space curves. *AMS Transl. 171* (1996), 11–22.
- [A 5] The geometry of spherical curves and the algebra of quaternions. *Russ. Math. Surv.* 50, No. 1 (1995), 3–68.
- [Bl1] BLASCHKE, W. Kreis und Kugel. Leipzig, 1916.
- [B12] Vorlesungen über Differentialgeometrie II. Springer-Verlag, 1923.
- [Bu] BUSEMAN, H. The foundations of Minkowskian geometry. *Comm. Math. Helv.* 24 (1950), 156–187.
- [Ge] GERICKE, H. Zur Relativ-Geometrie ebener Kurven. *Math. Zeitschr.* 47 (1942), 215–228.
- [Gu 1] GUGGENHEIMER, H. Sign changes, extrema, and curves of minimal order. J. Diff. Geom. 3 (1969), 511–521.
- [Gu 2] On plane Minkowski geometry. Geom. Dedicata 12 (1982), 371–381.
- [G-M-O] GUIEU, L., E. MOURRE and V. OVSIENKO. Theorem on six vertices of a plane curve via the Sturm theory. *The Arnold-Gelfand Mathematical Seminars*, Birkhäuser, 1997, 257–266.
- [G-O] GUIEU, L. and V. OVSIENKO. Caustique affine et caustique gravitationnelle d'une courbe plane. In preparation.
- [He 1] HEIL, E. Der Vierscheitelsatz in Relativ- und Minkowski-Geometrie. Monatsh. für Math. 74 (1970), 97–107.