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**Autor:** ECKMANN, Beno  
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## 2.2. EXAMPLES.

1) In [H-W] it is shown, by a simple argument, that  $q(\mathbf{Z}^n) \geq 0$  for all  $n \geq 1$ . We return to that case later on. Here we just recall that  $q(\mathbf{Z}) = q(\mathbf{Z}^2) = q(\mathbf{Z}^4) = 0$ , as is easily seen by taking an appropriate  $M$  with  $\chi(M) = 0$ . However for  $\mathbf{Z}^3$  one only gets  $0 \leq q \leq 2$ , the deficiency being 0.

2) For the surface group  $\Sigma_g$ ,  $g \geq 2$ , i.e. the fundamental group of the closed orientable surface of genus  $g$ , one has  $\text{def}(\Sigma_g) = 2g - 1$  and  $\beta_1 = 2g$ . Thus

$$2 - 4g \leq q(\Sigma_g) \leq 4 - 4g.$$

3) For any knot group  $G$  (the fundamental group of the complement of a classical knot in  $S^3$ ) the deficiency is 1 and  $\beta_1 = 1$  whence  $q(G) = 0$ .

4) Let  $G$  be a 2-knot-group, i.e. the fundamental group of the complement of two-dimensional knot  $S^2$  in  $S^4$ . As for classical knots  $\beta_1(G) = 1$ . Surgery along the imbedded sphere  $S^2$  produces a 4-manifold  $M$  with fundamental group  $G$ , and with  $\beta_2(M) = 0$ , whence  $\chi M = 0$ . Thus again  $q(G) = 0$ .

2.3. There is a topological ingredient available in 4-manifolds which has not been used, namely the signature. This has suggested a more refined group invariant associated with 4-manifolds, see the next section.

3. THE  $(\chi + \sigma)$ -INVARIANT

3.1. We recall that the cohomology group  $H^2(M; \mathbf{R})$  is a real quadratic space, the quadratic form being given by the cup-product evaluated on the fundamental cycle of  $M$ . It is non-degenerate, and the space splits into a positive-definite and a negative-definite subspace of dimensions  $\beta_2^+$  and  $\beta_2^-$  respectively. The difference  $\beta_2^+ - \beta_2^- = \sigma(M)$  is the signature of  $M$ . Its sign clearly depends on the orientation of  $M$  and we assume the orientation chosen in such a way that  $\sigma(M) \leq 0$ , i.e.,  $\beta_2^+ \leq \beta_2^-$ . Since  $\beta_2 = \beta_2^+ + \beta_2^-$  the sum  $\chi(M) + \sigma(M)$  is equal to  $2 - 2\beta_1(G) + 2\beta_2^+(M)$ , where as always  $G = \pi_1(M)$ . Since that sum is bounded below by  $2 - 2\beta_1(G)$  depending on  $G$  only one can define an invariant  $p(G)$  to be the minimum of  $\chi(M) + \sigma(M)$  for all  $M$  with fundamental group  $G$  and oriented in such a way that  $\sigma(M) \leq 0$ . Obviously  $p(G) \leq q(G)$ . An equivalent way to define  $p(G)$  is to take, independently of orientations, the minimum of  $\chi(M) - |\sigma(M)|$ .

Putting together all above inequalities we get

$$2 - 2\beta_1(G) \leq p(G) \leq q(G) \leq 2 - 2\text{def}(G).$$

3.2. It seems difficult in general to compute the value of  $p(G)$  and  $q(G)$ , and their group-theoretic meaning is not known. We first show how one can proceed in special cases where information on  $H^2(G)$ , i.e.  $H^2$  of the Eilenberg-MacLane space  $K(G, 1)$  is available. We then show (Section 3.3) that it is quite interesting for applications to know that the two invariants are non-negative. (This is clearly the case if  $\beta_1(G) \leq 1$ , in particular if  $G$  is finite).

Any 4-manifold  $M$  with  $\pi_1(M) = G$  can be imbedded in a  $K(G, 1)$  by adding cells of dimension  $2, 3, \dots$  in order to kill the homotopy groups in dimensions  $\geq 2$ . This yields an injective map  $H^2(G; \mathbf{R}) \rightarrow H^2(M; \mathbf{R})$ . If in  $H^2(G; \mathbf{R})$  the cup-product happens to be trivial then  $H^2(M; \mathbf{R})$  contains an isotropic subspace of dimension  $\beta_2(G)$ . In that case  $\beta_2^+(M)$  must be  $\geq \beta_2(G)$  so that

$$p(G) \geq 2 - 2\beta_1(G) + 2\beta_2(G).$$

This applies to examples in 2.2:

For the group  $G = \mathbf{Z}^3$  the 3-dimensional torus is a  $K(G, 1)$  and the cup-product in  $H^2$  is trivial. Since  $\beta_1(G) = \beta_2(G) = 3$  we get  $p(\mathbf{Z}^3) \geq 2$  whence  $p(\mathbf{Z}^3) = q(\mathbf{Z}^3) = 2$ .

For  $G = \Sigma_g$ ,  $g \geq 2$ , the surface of genus  $g$  is a  $K(G, 1)$ , and  $\beta_1(G) = 2g$ ,  $\beta_2(G) = 1$ . Thus  $p(G) \geq 4 - 4g$  whence  $p(\Sigma_g) = q(\Sigma_g) = 4 - 4g$ . So here the invariants are negative. Another such case is the free group  $F_m$  on  $m \geq 2$  generators where one easily finds  $p(F_m) = q(F_m) = 2 - 2m$ .

3.3. There are several instances where the sign of the invariants yields significant information on the 4-manifolds or the groups involved. We mention three of them.

I) *Deficiency*. From the inequality in 2.1 one immediately notes that if  $q(G) \geq 0$  then  $\text{def}(G) \leq 1$ . We will return to this fact later on.

II) *Complex surfaces*. We assume that our 4-manifold  $M$  is a complex surface (complex dimension 2). Then it is known that  $\chi + \sigma$  of  $M$  can be expressed in different ways: We write  $c_2$  for the second Chern class  $c_2(M)$  evaluated on  $M$ ,  $c_1^2$  for the cup-square of the first Chern class evaluated on  $M$ . Then  $\chi(M) = c_2$  and  $\sigma(M) = 1/3(c_1^2 - 2c_2)$  [since the signature is  $1/3$  of the first Pontrjagin number, which in the complex case can be expressed by the Chern classes as above]. Thus

$$\chi(M) + \sigma(M) = c_2 + 1/3(c_1^2 - 2c_2) = 1/3(c_1^2 + c_2).$$

This is 4 times the holomorphic Euler characteristic  $1 - g_1 + g_2$  of  $M$  by the Riemann-Roch theorem.

PROPOSITION 1. *Let  $M$  be a complex surface, and assume that its fundamental group  $G$  fulfills  $p(G) \geq 0$ . Then the holomorphic Euler characteristic of  $M$  is  $\geq 0$ .*

By the Kodaira-Enriques classification it follows that  $M$  cannot be ruled over a curve of genus  $\geq 2$ .

REMARK. The formulae above leading to the holomorphic Euler characteristic refer to the orientation of the complex surface dictated by the complex structure. Thus the argument is valid only if in *that* orientation  $\sigma(M) \leq 0$ . If however  $\sigma(M) > 0$  then  $p(G) \geq 0$  implies that  $2 - 2\beta_1(G) + 2\beta_2^+_{\text{wrong}}(M) \geq 0$  where  $\beta_2^+_{\text{wrong}}$  refers to the “wrong” orientation and is  $= \beta_2^-(M)$ . Now  $\beta_2^+(M) > \beta_2^-(M)$  by assumption. Thus the result remains true; the holomorphic characteristic is  $> 0$ .

III) *Donaldson Theory*. Finitely presented groups  $G$  with  $p(G) \geq 0$  and  $\beta_1(G) \geq 4$  do not qualify for the Theorems A, B, and C of Donaldson [D] relating to non-simply connected topological manifolds. Indeed in these theorems the signature is assumed to be negative with  $\beta_2^+ = 0, 1$  or  $2$ . However  $p(G) \geq 0$  means  $2 - 2\beta_1(G) + 2\beta_2^+(M) \geq 0$ , i.e.  $\beta_2^+(M) \geq \beta_1(G) - 1$ .

#### 4. DEUS EX MACHINA: $l_2$ -COHOMOLOGY

4.1. We recall in a few words the (cellular) definition of  $l_2$ -cohomology and  $l_2$ -Betti numbers, in the case of a 4-manifold  $M$  but things apply to any finite cell-complex.

Some definitions: For any countable group  $G$  let  $l_2G$  be the Hilbert space of square-integrable real functions on  $G$ , with  $G$  operating on the left, and  $NG$  the algebra of bounded  $G$ -equivariant linear operators on  $l_2G$ . A Hilbert- $G$ -module  $H$  is a Hilbert space with isometric left  $G$ -action which admits an isometric  $G$ -equivariant imbedding into some  $l_2G^m$  (direct sum of  $m$  copies of  $l_2G$ ). The projection operator  $\phi$  of  $l_2G^m$  with image  $H$  is given by a matrix  $(\phi_{kl})$ ,  $\phi_{kl} \in NG$ . The “trace”  $\sum \langle \phi_{kk}(1), 1 \rangle$  is the von Neumann dimension  $\dim_G H$ ; it is a real number  $\geq 0$ , and  $= 0$  if and only if  $H = 0$ .

Let  $\tilde{M}$  be the universal cover of  $M$  with the cell-decomposition corresponding to that chosen in  $M$ . The square-integrable real  $i$ -cochains of  $\tilde{M}$  constitute a Hilbert space  $C_{(2)}^i(\tilde{M})$  with isometric  $G$ -action. It decomposes into the direct sum of  $\alpha_i$  copies of  $l_2G$ ,  $i = 0, \dots, 4$ . As before  $\alpha_i$  denotes the