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Artikel: 4-MANIFOLDS, GROUP INVARIANTS, AND l_2-BETTI NUMBERS
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2.2. EXAMPLES.

1) In [H-W] it is shown, by a simple argument, that $q(\mathbf{Z}^n) \geq 0$ for all $n \geq 1$. We return to that case later on. Here we just recall that $q(\mathbf{Z}) = q(\mathbf{Z}^2) = q(\mathbf{Z}^4) = 0$, as is easily seen by taking an appropriate M with $\chi(M) = 0$. However for $\mathbf{Z}^3$ one only gets $0 \leq q \leq 2$, the deficiency being 0.

2) For the surface group Σ_g, $g \geq 2$, i.e. the fundamental group of the closed orientable surface of genus g, one has $\mathrm{def}(\Sigma_g) = 2g - 1$ and $\beta_1 = 2g$. Thus

$$2 - 4g \leq q(\Sigma_g) \leq 4 - 4g\,.$$

3) For any knot group G (the fundamental group of the complement of a classical knot in S^3) the deficiency is 1 and $\beta_1 = 1$ whence $q(G) = 0$.

4) Let G be a 2-knot-group, i.e. the fundamental group of the complement of two-dimensional knot S^2 in S^4. As for classical knots $\beta_1(G) = 1$. Surgery along the imbedded sphere S^2 produces a 4-manifold M with fundamental group G, and with $\beta_2(M) = 0$, whence $\chi M = 0$. Thus again $q(G) = 0$.

2.3. There is a topological ingredient available in 4-manifolds which has not been used, namely the signature. This has suggested a more refined group invariant associated with 4-manifolds, see the next section.

3. THE $(\chi + \sigma)$-INVARIANT

3.1. We recall that the cohomology group $H^2(M; \mathbf{R})$ is a real quadratic space, the quadratic form being given by the cup-product evaluated on the fundamental cycle of M. It is non-degenerate, and the space splits into a positive-definite and a negative-definite subspace of dimensions β_2^+ and β_2^- respectively. The difference $\beta_2^+ - \beta_2^- = \sigma(M)$ is the signature of M. Its sign clearly depends on the orientation of M and we assume the orientation chosen in such a way that $\sigma(M) \leq 0$, i.e., $\beta_2^+ \leq \beta_2^-$. Since $\beta_2 = \beta_2^+ + \beta_2^-$ the sum $\chi(M) + \sigma(M)$ is equal to $2 - 2\beta_1(G) + 2\beta_2^+(M)$, where as always $G = \pi_1(M)$. Since that sum is bounded below by $2 - 2\beta_1(G)$ depending on G only one can define an invariant $p(G)$ to be the minimum of $\chi(M) + \sigma(M)$ for all M with fundamental group G and oriented in such a way that $\sigma(M) \leq 0$. Obviously $p(G) \leq q(G)$. An equivalent way to define $p(G)$ is to take, independently of orientations, the minimum of $\chi(M) - |\sigma(M)|$.

Putting together all above inequalities we get

$$2 - 2\beta_1(G) \leq p(G) \leq q(G) \leq 2 - 2\,\mathrm{def}(G)\,.$$

3.2. It seems difficult in general to compute the value of $p(G)$ and $q(G)$, and their group-theoretic meaning is not known. We first show how one can proceed in special cases where information on $H^2(G)$, i.e. H^2 of the Eilenberg-MacLane space $K(G,1)$ is available. We then show (Section 3.3) that it is quite interesting for applications to know that the two invariants are non-negative. (This is clearly the case if $\beta_1(G) \leq 1$, in particular if G is finite).

Any 4-manifold M with $\pi_1(M) = G$ can be imbedded in a $K(G,1)$ by adding cells of dimension $2,3,\dots$ in order to kill the homotopy groups in dimensions ≥ 2. This yields an injective map $H^2(G;\mathbf{R}) \longrightarrow H^2(M;\mathbf{R})$. If in $H^2(G;\mathbf{R})$ the cup-product happens to be trivial then $H^2(M;\mathbf{R})$ contains an isotropic subspace of dimension $\beta_2(G)$. In that case $\beta_2^+(M)$ must be $\geq \beta_2(G)$ so that

$$p(G) \geq 2 - 2\beta_1(G) + 2\beta_2(G).$$

This applies to examples in 2.2:

For the group $G = \mathbf{Z}^3$ the 3-dimensional torus is a $K(G,1)$ and the cup-product in H^2 is trivial. Since $\beta_1(G) = \beta_2(G) = 3$ we get $p(\mathbf{Z}^3) \geq 2$ whence $p(\mathbf{Z}^3) = q(\mathbf{Z}^3) = 2$.

For $G = \Sigma_g$, $g \geq 2$, the surface of genus g is a $K(G,1)$, and $\beta_1(G) = 2g$, $\beta_2(G) = 1$. Thus $p(G) \geq 4-4g$ whence $p(\Sigma_g) = q(\Sigma_g) = 4-4g$. So here the invariants are negative. Another such case is the free group F_m on $m \geq 2$ generators where one easily finds $p(F_m) = q(F_m) = 2 - 2m$.

3.3. There are several instances where the sign of the invariants yields significant information on the 4-manifolds or the groups involved. We mention three of them.

I) *Deficiency.* From the inequality in 2.1 one immediately notes that if $q(G) \geq 0$ then $\mathrm{def}(G) \leq 1$. We will return to this fact later on.

II) *Complex surfaces.* We assume that our 4-manifold M is a complex surface (complex dimension 2). Then it is known that $\chi + \sigma$ of M can be expressed in different ways: We write c_2 for the second Chern class $c_2(M)$ evaluated on M, c_1^2 for the cup-square of the first Chern class evaluated on M. Then $\chi(M) = c_2$ and $\sigma(M) = 1/3(c_1^2 - 2c_2)$ [since the signature is 1/3 of the first Pontrjagin number, which in the complex case can be expressed by the Chern classes as above]. Thus

$$\chi(M) + \sigma(M) = c_2 + 1/3(c_1^2 - 2c_2) = 1/3(c_1^2 + c_2).$$

This is 4 times the holomorphic Euler characteristic $1 - g_1 + g_2$ of M by the Riemann-Roch theorem.

PROPOSITION 1. *Let M be a complex surface, and assume that its fundamental group G fulfills $p(G) \geq 0$. Then the holomorphic Euler characteristic of M is ≥ 0.*

By the Kodaira-Enriques classification it follows that M cannot be ruled over a curve of genus ≥ 2.

REMARK. The formulae above leading to the holomorphic Euler characteristic refer to the orientation of the complex surface dictated by the complex structure. Thus the argument is valid only if in *that* orientation $\sigma(M) \leq 0$. If however $\sigma(M) > 0$ then $p(G) \geq 0$ implies that $2 - 2\beta_1(G) + 2\beta_{2\ \mathrm{wrong}}^{+}(M) \geq 0$ where $\beta_{2\ \mathrm{wrong}}^{+}$ refers to the "wrong" orientation and is $= \beta_2^-(M)$. Now $\beta_2^+(M) > \beta_2^-(M)$ by assumption. Thus the result remains true; the holomorphic characteristic is > 0.

III) *Donaldson Theory.* Finitely presented groups G with $p(G) \geq 0$ and $\beta_1(G) \geq 4$ do not qualify for the Theorems A,B, and C of Donaldson [D] relating to non-simply connected topological manifolds. Indeed in these theorems the signature is assumed to be negative with $\beta_2^+ = 0$, 1 or 2. However $p(G) \geq 0$ means $2 - 2\beta_1(G) + 2\beta_2^+(M) \geq 0$, i.e. $\beta_2^+(M) \geq \beta_1(G) - 1$.

4. DEUS EX MACHINA: l_2-COHOMOLOGY

4.1. We recall in a few words the (cellular) definition of l_2-cohomology and l_2-Betti numbers, in the case of a 4-manifold M but things apply to any finite cell-complex.

Some definitions: For any countable group G let l_2G be the Hilbert space of square-integrable real functions on G, with G operating on the left, and NG the algebra of bounded G-equivariant linear operators on l_2G. A Hilbert-G-module H is a Hilbert space with isometric left G-action which admits an isometric G-equivariant imbedding into some l_2G^m (direct sum of m copies of l_2G). The projection operator ϕ of l_2G^m with image H is given by a matrix (ϕ_{kl}), $\phi_{kl} \in NG$. The "trace" $\sum \langle \phi_{kk}(1), 1 \rangle$ is the von Neumann dimension $\dim_G H$; it is a real number ≥ 0, and $= 0$ if and only if $H = 0$.

Let $\widetilde{M}$ be the universal cover of M with the cell-decomposition corresponding to that chosen in M. The square-integrable real i-cochains of $\widetilde{M}$ constitute a Hilbert space $C_{(2)}^i(\widetilde{M})$ with isometric G-action. It decomposes into the direct sum of α_i copies of l_2G, $i = 0, \ldots, 4$. As before α_i denotes the