Zeitschrift: L'Enseignement Mathématique

Herausgeber: Commission Internationale de l'Enseignement Mathématique

**Band:** 43 (1997)

Heft: 3-4: L'ENSEIGNEMENT MATHÉMATIQUE

**Artikel:** 4-MANIFOLDS, GROUP INVARIANTS, AND \$1\_2\$-BETTI NUMBERS

Autor: ECKMANN, Beno
Kapitel: 3. The (+) -invariant

**DOI:** https://doi.org/10.5169/seals-63279

### Nutzungsbedingungen

Die ETH-Bibliothek ist die Anbieterin der digitalisierten Zeitschriften auf E-Periodica. Sie besitzt keine Urheberrechte an den Zeitschriften und ist nicht verantwortlich für deren Inhalte. Die Rechte liegen in der Regel bei den Herausgebern beziehungsweise den externen Rechteinhabern. Das Veröffentlichen von Bildern in Print- und Online-Publikationen sowie auf Social Media-Kanälen oder Webseiten ist nur mit vorheriger Genehmigung der Rechteinhaber erlaubt. Mehr erfahren

### **Conditions d'utilisation**

L'ETH Library est le fournisseur des revues numérisées. Elle ne détient aucun droit d'auteur sur les revues et n'est pas responsable de leur contenu. En règle générale, les droits sont détenus par les éditeurs ou les détenteurs de droits externes. La reproduction d'images dans des publications imprimées ou en ligne ainsi que sur des canaux de médias sociaux ou des sites web n'est autorisée qu'avec l'accord préalable des détenteurs des droits. En savoir plus

#### Terms of use

The ETH Library is the provider of the digitised journals. It does not own any copyrights to the journals and is not responsible for their content. The rights usually lie with the publishers or the external rights holders. Publishing images in print and online publications, as well as on social media channels or websites, is only permitted with the prior consent of the rights holders. Find out more

**Download PDF:** 03.12.2025

ETH-Bibliothek Zürich, E-Periodica, https://www.e-periodica.ch

## 2.2. EXAMPLES.

- 1) In [H-W] it is shown, by a simple argument, that  $q(\mathbf{Z}^n) \geq 0$  for all  $n \geq 1$ . We return to that case later on. Here we just recall that  $q(\mathbf{Z}) = q(\mathbf{Z}^2) = q(\mathbf{Z}^4) = 0$ , as is easily seen by taking an appropriate M with  $\chi(M) = 0$ . However for  $\mathbf{Z}^3$  one only gets  $0 \leq q \leq 2$ , the deficiency being 0.
- 2) For the surface group  $\Sigma_g$ ,  $g \geq 2$ , i.e. the fundamental group of the closed orientable surface of genus g, one has  $\operatorname{def}(\Sigma_g) = 2g 1$  and  $\beta_1 = 2g$ . Thus

$$2-4g\leq q(\Sigma_g)\leq 4-4g.$$

- 3) For any knot group G (the fundamental group of the complement of a classical knot in  $S^3$ ) the deficiency is 1 and  $\beta_1 = 1$  whence q(G) = 0.
- 4) Let G be a 2-knot-group, i.e. the fundamental group of the complement of two-dimensional knot  $S^2$  in  $S^4$ . As for classical knots  $\beta_1(G) = 1$ . Surgery along the imbedded sphere  $S^2$  produces a 4-manifold M with fundamental group G, and with  $\beta_2(M) = 0$ , whence  $\chi M = 0$ . Thus again q(G) = 0.
- 2.3. There is a topological ingredient available in 4-manifolds which has not been used, namely the signature. This has suggested a more refined group invariant associated with 4-manifolds, see the next section.

# 3. The $(\chi + \sigma)$ -invariant

3.1. We recall that the cohomology group  $H^2(M; \mathbf{R})$  is a real quadratic space, the quadratic form being given by the cup-product evaluated on the fundamental cycle of M. It is non-degenerate, and the space splits into a positive-definite and a negative-definite subspace of dimensions  $\beta_2^+$  and  $\beta_2^-$  respectively. The difference  $\beta_2^+ - \beta_2^- = \sigma(M)$  is the signature of M. Its sign clearly depends on the orientation of M and we assume the orientation chosen in such a way that  $\sigma(M) \leq 0$ , i.e.,  $\beta_2^+ \leq \beta_2^-$ . Since  $\beta_2 = \beta_2^+ + \beta_2^-$  the sum  $\chi(M) + \sigma(M)$  is equal to  $2 - 2\beta_1(G) + 2\beta_2^+(M)$ , where as always  $G = \pi_1(M)$ . Since that sum is bounded below by  $2 - 2\beta_1(G)$  depending on G only one can define an invariant  $\rho(G)$  to be the minimum of  $\chi(M) + \sigma(M)$  for all M with fundamental group G and oriented in such a way that  $\sigma(M) \leq 0$ . Obviously  $\rho(G) \leq q(G)$ . An equivalent way to define  $\rho(G)$  is to take, independently of orientations, the minimum of  $\chi(M) - |\sigma(M)|$ .

Putting together all above inequalities we get

$$2 - 2\beta_1(G) \le p(G) \le q(G) \le 2 - 2 \operatorname{def}(G)$$
.

3.2. It seems difficult in general to compute the value of p(G) and q(G), and their group-theoretic meaning is not known. We first show how one can proceed in special cases where information on  $H^2(G)$ , i.e.  $H^2$  of the Eilenberg-MacLane space K(G,1) is available. We then show (Section 3.3) that it is quite interesting for applications to know that the two invariants are non-negative. (This is clearly the case if  $\beta_1(G) \leq 1$ , in particular if G is finite).

Any 4-manifold M with  $\pi_1(M) = G$  can be imbedded in a K(G,1) by adding cells of dimension  $2,3,\ldots$  in order to kill the homotopy groups in dimensions  $\geq 2$ . This yields an injective map  $H^2(G;\mathbf{R}) \longrightarrow H^2(M;\mathbf{R})$ . If in  $H^2(G;\mathbf{R})$  the cup-product happens to be trivial then  $H^2(M;\mathbf{R})$  contains an isotropic subspace of dimension  $\beta_2(G)$ . In that case  $\beta_2^+(M)$  must be  $\geq \beta_2(G)$  so that

$$p(G) \ge 2 - 2\beta_1(G) + 2\beta_2(G)$$
.

This applies to examples in 2.2:

For the group  $G = \mathbb{Z}^3$  the 3-dimensional torus is a K(G, 1) and the cupproduct in  $H^2$  is trivial. Since  $\beta_1(G) = \beta_2(G) = 3$  we get  $p(\mathbb{Z}^3) \geq 2$  whence  $p(\mathbb{Z}^3) = q(\mathbb{Z}^3) = 2$ .

For  $G = \Sigma_g$ ,  $g \ge 2$ , the surface of genus g is a K(G,1), and  $\beta_1(G) = 2g$ ,  $\beta_2(G) = 1$ . Thus  $p(G) \ge 4-4g$  whence  $p(\Sigma_g) = q(\Sigma_g) = 4-4g$ . So here the invariants are negative. Another such case is the free group  $F_m$  on  $m \ge 2$  generators where one easily finds  $p(F_m) = q(F_m) = 2 - 2m$ .

- 3.3. There are several instances where the sign of the invariants yields significant information on the 4-manifolds or the groups involved. We mention three of them.
- I) Deficiency. From the inequality in 2.1 one immediately notes that if  $q(G) \ge 0$  then  $def(G) \le 1$ . We will return to this fact later on.
- II) Complex surfaces. We assume that our 4-manifold M is a complex surface (complex dimension 2). Then it is known that  $\chi + \sigma$  of M can be expressed in different ways: We write  $c_2$  for the second Chern class  $c_2(M)$  evaluated on M,  $c_1^2$  for the cup-square of the first Chern class evaluated on M. Then  $\chi(M) = c_2$  and  $\sigma(M) = 1/3(c_1^2 2c_2)$  [since the signature is 1/3 of the first Pontrjagin number, which in the complex case can be expressed by the Chern classes as above]. Thus

$$\chi(M) + \sigma(M) = c_2 + 1/3(c_1^2 - 2c_2) = 1/3(c_1^2 + c_2).$$

This is 4 times the holomorphic Euler characteristic  $1 - g_1 + g_2$  of M by the Riemann-Roch theorem.

PROPOSITION 1. Let M be a complex surface, and assume that its fundamental group G fulfills  $p(G) \ge 0$ . Then the holomorphic Euler characteristic of M is  $\ge 0$ .

By the Kodaira-Enriques classification it follows that M cannot be ruled over a curve of genus  $\geq 2$ .

REMARK. The formulae above leading to the holomorphic Euler characteristic refer to the orientation of the complex surface dictated by the complex structure. Thus the argument is valid only if in *that* orientation  $\sigma(M) \leq 0$ . If however  $\sigma(M) > 0$  then  $p(G) \geq 0$  implies that  $2 - 2\beta_1(G) + 2\beta_2^+_{\text{wrong}}(M) \geq 0$  where  $\beta_2^+_{\text{wrong}}$  refers to the "wrong" orientation and is  $= \beta_2^-(M)$ . Now  $\beta_2^+(M) > \beta_2^-(M)$  by assumption. Thus the result remains true; the holomorphic characteristic is > 0.

III) Donaldson Theory. Finitely presented groups G with  $p(G) \ge 0$  and  $\beta_1(G) \ge 4$  do not qualify for the Theorems A,B, and C of Donaldson [D] relating to non-simply connected topological manifolds. Indeed in these theorems the signature is assumed to be negative with  $\beta_2^+ = 0$ , 1 or 2. However  $p(G) \ge 0$  means  $2 - 2\beta_1(G) + 2\beta_2^+(M) \ge 0$ , i.e.  $\beta_2^+(M) \ge \beta_1(G) - 1$ .

# 4. Deus ex machina: $l_2$ -cohomology

4.1. We recall in a few words the (cellular) definition of  $l_2$ -cohomology and  $l_2$ -Betti numbers, in the case of a 4-manifold M but things apply to any finite cell-complex.

Some definitions: For any countable group G let  $l_2G$  be the Hilbert space of square-integrable real functions on G, with G operating on the left, and NG the algebra of bounded G-equivariant linear operators on  $l_2G$ . A Hilbert-G-module H is a Hilbert space with isometric left G-action which admits an isometric G-equivariant imbedding into some  $l_2G^m$  (direct sum of m copies of  $l_2G$ ). The projection operator  $\phi$  of  $l_2G^m$  with image H is given by a matrix  $(\phi_{kl})$ ,  $\phi_{kl} \in NG$ . The "trace"  $\sum \langle \phi_{kk}(1), 1 \rangle$  is the von Neumann dimension  $\dim_G H$ ; it is a real number  $\geq 0$ , and = 0 if and only if H = 0.

Let  $\widetilde{M}$  be the universal cover of M with the cell-decomposition corresponding to that chosen in M. The square-integrable real i-cochains of  $\widetilde{M}$  constitute a Hilbert space  $C^i_{(2)}(\widetilde{M})$  with isometric G-action. It decomposes into the direct sum of  $\alpha_i$  copies of  $l_2G$ ,  $i=0,\ldots,4$ . As before  $\alpha_i$  denotes the