Zeitschrift: L'Enseignement Mathématique
Herausgeber: Commission Internationale de l'Enseignement Mathématique
Band: 43 (1997)
Heft: 3-4: L'ENSEIGNEMENT MATHÉMATIQUE

Artikel: 4-MANIFOLDS, GROUP INVARIANTS, AND l_2-BETTI NUMBERS
Autor: ECKMANN, Beno
Kapitel: 2. The Hausmann-Weinberger invariant
DOI: https://doi.org/10.5169/seals-63279

boundary yields a new 4-manifold where the element corresponding to r_1 has been killed; and similarly for the other r_i. Let M_0 be the 4-manifold thus obtained, fulfilling $\pi_1(M_0) = G$. The idea of that construction can already be found in the old book [S-T]. Much later the procedure, in a more general context, has been called "elementary surgery".

1.2. We recall that the (good old) Euler characteristic $\chi(X)$ of a finite cell complex X is the alternating sum

$$\chi(X) = \sum (-1)^i \alpha_i ,$$

where α_i is the number of i-cells. It is easily computed for M_0 above: For M' it is $2 - 2m$ since it is $= 0$ for $S^1 \times S^3$ and since it decreases by 2 in a connected sum. Under the surgery process above it increases by 2 [use the fact that for the union of two complexes X and Y with intersection Z the characteristic is $\chi(X) + \chi(Y) - \chi(Z)$; and that $\chi(B^2 \times S^2) = 2$]. Whence

$$\chi(M_0) = 2 - 2m + 2n = 2 - 2(m - n) .$$

The difference $m - n$ is called the deficiency of the presentation of G.

1.3. On the other hand the characteristic can be expressed by the Betti numbers of the cell complex X as $\sum (-1)^i \beta_i(X)$ where $\beta_i(X) = \dim_{\mathbf{R}} H_i(X; \mathbf{R})$ (and is therefore a topological invariant). Moreover the β_i of a manifold fulfill Poincaré duality, i.e. they are equal in complementary dimensions. Thus $\chi(M) = 2 - 2\beta_1(M) + \beta_2(M)$. We recall that homology in dimension 1 depends on the fundamental group G only; β_1 is the $\mathbf{Q}$-rank of G Abelianised and we write $\beta_1(G)$ for $\beta_1(M)$. Comparing with $\chi(M_0)$ above we see that the deficiency of the presentation is $\leq \beta_1(G)$. Thus there is a maximum for the deficiency of all presentations of G, called the deficiency $\operatorname{def}(G)$ of G. [For this simple side result there are, of course, much easier arguments.]

2. The Hausmann-Weinberger invariant

2.1. As seen above, the Euler characteristic of a 4-manifold M with given finitely presented fundamental group G is bounded below by $2 - 2\beta_1(G)$. The minimum of $\chi(M)$ for all such M has been considered by Hausmann-Weinberger [H-W] and denoted by $q(G)$. Using M_0 above we have the inequalities

$$2 - 2\beta_1(G) \leq q(G) \leq 2 - 2\operatorname{def}(G) .$$

2.2. EXAMPLES.

1) In [H-W] it is shown, by a simple argument, that $q(\mathbf{Z}^n) \geq 0$ for all $n \geq 1$. We return to that case later on. Here we just recall that $q(\mathbf{Z}) = q(\mathbf{Z}^2) = q(\mathbf{Z}^4) = 0$, as is easily seen by taking an appropriate M with $\chi(M) = 0$. However for $\mathbf{Z}^3$ one only gets $0 \leq q \leq 2$, the deficiency being 0.

2) For the surface group Σ_g, $g \geq 2$, i.e. the fundamental group of the closed orientable surface of genus g, one has $\operatorname{def}(\Sigma_g) = 2g - 1$ and $\beta_1 = 2g$. Thus

$$2 - 4g \leq q(\Sigma_g) \leq 4 - 4g .$$

3) For any knot group G (the fundamental group of the complement of a classical knot in S^3) the deficiency is 1 and $\beta_1 = 1$ whence $q(G) = 0$.

4) Let G be a 2-knot-group, i.e. the fundamental group of the complement of two-dimensional knot S^2 in S^4. As for classical knots $\beta_1(G) = 1$. Surgery along the imbedded sphere S^2 produces a 4-manifold M with fundamental group G, and with $\beta_2(M) = 0$, whence $\chi M = 0$. Thus again $q(G) = 0$.

2.3. There is a topological ingredient available in 4-manifolds which has not been used, namely the signature. This has suggested a more refined group invariant associated with 4-manifolds, see the next section.

3. THE $(\chi + \sigma)$-INVARIANT

3.1. We recall that the cohomology group $H^2(M; \mathbf{R})$ is a real quadratic space, the quadratic form being given by the cup-product evaluated on the fundamental cycle of M. It is non-degenerate, and the space splits into a positive-definite and a negative-definite subspace of dimensions β_2^+ and β_2^- respectively. The difference $\beta_2^+ - \beta_2^- = \sigma(M)$ is the signature of M. Its sign clearly depends on the orientation of M and we assume the orientation chosen in such a way that $\sigma(M) \leq 0$, i.e., $\beta_2^+ \leq \beta_2^-$. Since $\beta_2 = \beta_2^+ + \beta_2^-$ the sum $\chi(M) + \sigma(M)$ is equal to $2 - 2\beta_1(G) + 2\beta_2^+(M)$, where as always $G = \pi_1(M)$. Since that sum is bounded below by $2 - 2\beta_1(G)$ depending on G only one can define an invariant $p(G)$ to be the minimum of $\chi(M) + \sigma(M)$ for all M with fundamental group G and oriented in such a way that $\sigma(M) \leq 0$. Obviously $p(G) \leq q(G)$. An equivalent way to define $p(G)$ is to take, independently of orientations, the minimum of $\chi(M) - |\sigma(M)|$.

Putting together all above inequalities we get

$$2 - 2\beta_1(G) \leq p(G) \leq q(G) \leq 2 - 2\operatorname{def}(G) .$$