Zeitschrift: L'Enseignement Mathématique
Herausgeber: Commission Internationale de l'Enseignement Mathématique
Band: 43 (1997)
Heft: 3-4: L'ENSEIGNEMENT MATHÉMATIQUE

Artikel: 4-MANIFOLDS, GROUP INVARIANTS, AND l_2-BETTI NUMBERS
Autor: ECKMANN, Beno
Kapitel: 2. The Hausmann-Weinberger invariant
DOI: https://doi.org/10.5169/seals-63279

Nutzungsbedingungen
Die ETH-Bibliothek ist die Anbieterin der digitalisierten Zeitschriften auf E-Periodica. Sie besitzt keine Urheberrechte an den Zeitschriften und ist nicht verantwortlich für deren Inhalte. Die Rechte liegen in der Regel bei den Herausgebern beziehungsweise den externen Rechteinhabern. Das Veröffentlichen von Bildern in Print- und Online-Publikationen sowie auf Social Media-Kanälen oder Webseiten ist nur mit vorheriger Genehmigung der Rechteinhaber erlaubt. Mehr erfahren

Conditions d'utilisation
L'ETH Library est le fournisseur des revues numérisées. Elle ne détient aucun droit d'auteur sur les revues et n'est pas responsable de leur contenu. En règle générale, les droits sont détenus par les éditeurs ou les détenteurs de droits externes. La reproduction d'images dans des publications imprimées ou en ligne ainsi que sur des canaux de médias sociaux ou des sites web n'est autorisée qu'avec l'accord préalable des détenteurs des droits. En savoir plus

Terms of use
The ETH Library is the provider of the digitised journals. It does not own any copyrights to the journals and is not responsible for their content. The rights usually lie with the publishers or the external rights holders. Publishing images in print and online publications, as well as on social media channels or websites, is only permitted with the prior consent of the rights holders. Find out more

Download PDF: 15.01.2026

ETH-Bibliothek Zürich, E-Periodica, https://www.e-periodica.ch

boundary yields a new 4-manifold where the element corresponding to r_1 has been killed; and similarly for the other r_i. Let M_0 be the 4-manifold thus obtained, fulfilling $\pi_1(M_0) = G$. The idea of that construction can already be found in the old book [S-T]. Much later the procedure, in a more general context, has been called "elementary surgery".

1.2. We recall that the (good old) Euler characteristic $\chi(X)$ of a finite cell complex X is the alternating sum

$$\chi(X) = \sum (-1)^i \alpha_i ,$$

where α_i is the number of i-cells. It is easily computed for M_0 above: For M' it is $2 - 2m$ since it is $= 0$ for $S^1 \times S^3$ and since it decreases by 2 in a connected sum. Under the surgery process above it increases by 2 [use the fact that for the union of two complexes X and Y with intersection Z the characteristic is $\chi(X) + \chi(Y) - \chi(Z)$; and that $\chi(B^2 \times S^2) = 2$]. Whence

$$\chi(M_0) = 2 - 2m + 2n = 2 - 2(m - n) .$$

The difference $m - n$ is called the deficiency of the presentation of G.

1.3. On the other hand the characteristic can be expressed by the Betti numbers of the cell complex X as $\sum (-1)^i \beta_i(X)$ where $\beta_i(X) = \dim_{\mathbf{R}} H_i(X; \mathbf{R})$ (and is therefore a topological invariant). Moreover the β_i of a manifold fulfill Poincaré duality, i.e. they are equal in complementary dimensions. Thus $\chi(M) = 2 - 2\beta_1(M) + \beta_2(M)$. We recall that homology in dimension 1 depends on the fundamental group G only; β_1 is the $\mathbf{Q}$-rank of G Abelianised and we write $\beta_1(G)$ for $\beta_1(M)$. Comparing with $\chi(M_0)$ above we see that the deficiency of the presentation is $\leq \beta_1(G)$. Thus there is a maximum for the deficiency of all presentations of G, called the deficiency $\operatorname{def}(G)$ of G. [For this simple side result there are, of course, much easier arguments.]

2. The Hausmann-Weinberger invariant

2.1. As seen above, the Euler characteristic of a 4-manifold M with given finitely presented fundamental group G is bounded below by $2 - 2\beta_1(G)$. The minimum of $\chi(M)$ for all such M has been considered by Hausmann-Weinberger [H-W] and denoted by $q(G)$. Using M_0 above we have the inequalities

$$2 - 2\beta_1(G) \leq q(G) \leq 2 - 2\operatorname{def}(G) .$$

2.2. EXAMPLES.

1) In [H-W] it is shown, by a simple argument, that $q(\mathbf{Z}^n) \geq 0$ for all $n \geq 1$. We return to that case later on. Here we just recall that $q(\mathbf{Z}) = q(\mathbf{Z}^2) = q(\mathbf{Z}^4) = 0$, as is easily seen by taking an appropriate M with $\chi(M) = 0$. However for $\mathbf{Z}^3$ one only gets $0 \leq q \leq 2$, the deficiency being 0.

2) For the surface group Σ_g, $g \geq 2$, i.e. the fundamental group of the closed orientable surface of genus g, one has $\operatorname{def}(\Sigma_g) = 2g - 1$ and $\beta_1 = 2g$. Thus

$$2 - 4g \leq q(\Sigma_g) \leq 4 - 4g\,.$$

3) For any knot group G (the fundamental group of the complement of a classical knot in S^3) the deficiency is 1 and $\beta_1 = 1$ whence $q(G) = 0$.

4) Let G be a 2-knot-group, i.e. the fundamental group of the complement of two-dimensional knot S^2 in S^4. As for classical knots $\beta_1(G) = 1$. Surgery along the imbedded sphere S^2 produces a 4-manifold M with fundamental group G, and with $\beta_2(M) = 0$, whence $\chi M = 0$. Thus again $q(G) = 0$.

2.3. There is a topological ingredient available in 4-manifolds which has not been used, namely the signature. This has suggested a more refined group invariant associated with 4-manifolds, see the next section.

3. THE $(\chi + \sigma)$-INVARIANT

3.1. We recall that the cohomology group $H^2(M;\mathbf{R})$ is a real quadratic space, the quadratic form being given by the cup-product evaluated on the fundamental cycle of M. It is non-degenerate, and the space splits into a positive-definite and a negative-definite subspace of dimensions β_2^+ and β_2^- respectively. The difference $\beta_2^+ - \beta_2^- = \sigma(M)$ is the signature of M. Its sign clearly depends on the orientation of M and we assume the orientation chosen in such a way that $\sigma(M) \leq 0$, i.e., $\beta_2^+ \leq \beta_2^-$. Since $\beta_2 = \beta_2^+ + \beta_2^-$ the sum $\chi(M) + \sigma(M)$ is equal to $2 - 2\beta_1(G) + 2\beta_2^+(M)$, where as always $G = \pi_1(M)$. Since that sum is bounded below by $2 - 2\beta_1(G)$ depending on G only one can define an invariant $p(G)$ to be the minimum of $\chi(M) + \sigma(M)$ for all M with fundamental group G and oriented in such a way that $\sigma(M) \leq 0$. Obviously $p(G) \leq q(G)$. An equivalent way to define $p(G)$ is to take, independently of orientations, the minimum of $\chi(M) - |\sigma(M)|$.

Putting together all above inequalities we get

$$2 - 2\beta_1(G) \leq p(G) \leq q(G) \leq 2 - 2\operatorname{def}(G)\,.$$