

Zeitschrift: L'Enseignement Mathématique
Herausgeber: Commission Internationale de l'Enseignement Mathématique
Band: 43 (1997)
Heft: 3-4: L'ENSEIGNEMENT MATHÉMATIQUE

Artikel: 4-MANIFOLDS, GROUP INVARIANTS, AND H_2 -BETTI NUMBERS
Autor: ECKMANN, Beno
Kapitel: 1. A BASIC CONSTRUCTION
DOI: <https://doi.org/10.5169/seals-63279>

Nutzungsbedingungen

Die ETH-Bibliothek ist die Anbieterin der digitalisierten Zeitschriften auf E-Periodica. Sie besitzt keine Urheberrechte an den Zeitschriften und ist nicht verantwortlich für deren Inhalte. Die Rechte liegen in der Regel bei den Herausgebern beziehungsweise den externen Rechteinhabern. Das Veröffentlichen von Bildern in Print- und Online-Publikationen sowie auf Social Media-Kanälen oder Webseiten ist nur mit vorheriger Genehmigung der Rechteinhaber erlaubt. [Mehr erfahren](#)

Conditions d'utilisation

L'ETH Library est le fournisseur des revues numérisées. Elle ne détient aucun droit d'auteur sur les revues et n'est pas responsable de leur contenu. En règle générale, les droits sont détenus par les éditeurs ou les détenteurs de droits externes. La reproduction d'images dans des publications imprimées ou en ligne ainsi que sur des canaux de médias sociaux ou des sites web n'est autorisée qu'avec l'accord préalable des détenteurs des droits. [En savoir plus](#)

Terms of use

The ETH Library is the provider of the digitised journals. It does not own any copyrights to the journals and is not responsible for their content. The rights usually lie with the publishers or the external rights holders. Publishing images in print and online publications, as well as on social media channels or websites, is only permitted with the prior consent of the rights holders. [Find out more](#)

Download PDF: 27.04.2026

ETH-Bibliothek Zürich, E-Periodica, <https://www.e-periodica.ch>

4-MANIFOLDS, GROUP INVARIANTS, AND l_2 -BETTI NUMBERS

by Beno ECKMANN

It has been known for some time that closed 4-manifolds provide, via the fundamental group and the Euler characteristic, interesting invariants for finitely presented groups. In this short survey we describe these and more refined invariants (using also the signature of the manifold), and explain some of their significance. The invariants are not easily calculated in general, but quite good information is obtained using l_2 -Betti numbers.

The topic has been developed by several authors, more or less independently. We mention Hausmann-Weinberger [H-W], Kotschick [K], Lück [L], and myself [E1], [E2]. The paper [K] contains a wealth of information on the invariants and further important references; the application of l_2 -Betti numbers appears in [E2] and in [L].

1. A BASIC CONSTRUCTION

1.1. We will always denote by M a connected orientable closed 4-manifold (compact without boundary) admitting a cell decomposition. The fundamental group $G = \pi_1(M)$ is finitely presented. Indeed, homotopy classes of loops can be represented by edge-polygons and null-homotopies of these by using 2-cells. Conversely, any finitely presented group G is the fundamental group of a closed 4-manifold. If

$$G = \langle g_1, \dots, g_m \mid r_1, \dots, r_n \rangle$$

is a presentation of G , there is a standard procedure for constructing such a manifold: One first puts $M' = S^1 \times S^3 + \dots + S^1 \times S^3$, connected sum, one copy for each generator g_i of G . Then $\pi_1(M')$ is a free group on generators g_1, \dots, g_m . A relator, say r_1 , is a word in the g_i and can be represented by a loop S^1 in M' .

A tubular neighbourhood $S^1 \times B^3$ of S^1 , where B^k is the k -dimensional ball, has boundary $S^1 \times S^2$. Replacing the interior by $B^2 \times S^2$ with the same

boundary yields a new 4-manifold where the element corresponding to r_1 has been killed; and similarly for the other r_i . Let M_0 be the 4-manifold thus obtained, fulfilling $\pi_1(M_0) = G$. The idea of that construction can already be found in the old book [S-T]. Much later the procedure, in a more general context, has been called "elementary surgery".

1.2. We recall that the (good old) Euler characteristic $\chi(X)$ of a finite cell complex X is the alternating sum

$$\chi(X) = \sum (-1)^i \alpha_i,$$

where α_i is the number of i -cells. It is easily computed for M_0 above: For M' it is $2 - 2m$ since it is $= 0$ for $S^1 \times S^3$ and since it decreases by 2 in a connected sum. Under the surgery process above it increases by 2 [use the fact that for the union of two complexes X and Y with intersection Z the characteristic is $\chi(X) + \chi(Y) - \chi(Z)$; and that $\chi(B^2 \times S^2) = 2$]. Whence

$$\chi(M_0) = 2 - 2m + 2n = 2 - 2(m - n).$$

The difference $m - n$ is called the deficiency of the presentation of G .

1.3. On the other hand the characteristic can be expressed by the Betti numbers of the cell complex X as $\sum (-1)^i \beta_i(X)$ where $\beta_i(X) = \dim_{\mathbf{R}} H_i(X; \mathbf{R})$ (and is therefore a topological invariant). Moreover the β_i of a manifold fulfill Poincaré duality, i.e. they are equal in complementary dimensions. Thus $\chi(M) = 2 - 2\beta_1(M) + \beta_2(M)$. We recall that homology in dimension 1 depends on the fundamental group G only; β_1 is the \mathbf{Q} -rank of G Abelianised and we write $\beta_1(G)$ for $\beta_1(M)$. Comparing with $\chi(M_0)$ above we see that the deficiency of the presentation is $\leq \beta_1(G)$. Thus there is a maximum for the deficiency of all presentations of G , called the deficiency $\text{def}(G)$ of G . [For this simple side result there are, of course, much easier arguments.]

2. THE HAUSMANN-WEINBERGER INVARIANT

2.1. As seen above, the Euler characteristic of a 4-manifold M with given finitely presented fundamental group G is bounded below by $2 - 2\beta_1(G)$. The minimum of $\chi(M)$ for all such M has been considered by Hausmann-Weinberger [H-W] and denoted by $q(G)$. Using M_0 above we have the inequalities

$$2 - 2\beta_1(G) \leq q(G) \leq 2 - 2 \text{def}(G).$$