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Artikel: QUATERNARY CUBIC FORMS AND PROJECTIVE ALGEBRAIC THREEFOLDS

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PROPOSITION 4. *The set of sextuples in $\mathcal{U}$ whose associated cubic surface is given by an equation which is not a (nondegenerate) Sylvestrian pentahedral form is the Zariski-closed subset* $\{f^* I_{40} = 0\}$.

Of course, a better understanding of the geometric meaning of the other invariants should allow to extend this result.

II. CUBIC FORMS OF PROJECTIVE THREEFOLDS

1. PRELIMINARIES

For the convenience of the reader, we have collected the crucial theorems which we will use in the construction of our examples.

1.1. *The Lefschetz Theorem on Hyperplane Sections.* We summarize Bertini's Theorem and Lefschetz' Theorem in:

THEOREM 5. *Let Y be a projective manifold, L a very ample line bundle on Y, and $X := Z(s)$ the zero-set of a general section $s \in H^0(X, L)$. Then X is a manifold (connected if* $\dim Y \geq 2$*), and the inclusion $\iota \colon X \hookrightarrow Y$ induces isomorphisms*

$$\iota^* : H^i(Y, \mathbf{Z}) \longrightarrow H^i(X, \mathbf{Z}), \qquad i = 1, \ldots, \dim Y - 2;$$
$$\iota_* : \pi_i(X) \longrightarrow \pi_i(Y), \qquad i = 1, \ldots, \dim Y - 2.$$

Proof. [La], Th. 3.6.7 & Th. 8.1.1. □

1.2. *Formulas for Blow Ups.* A very simple way to obtain a new manifold from a given one is the blow up in a point or along a smooth curve. The cup form behaves as follows (we will suppose for simplicity that $H^2(Y, \mathbf{Z})$ is without torsion):

THEOREM 6. i) *Let $\sigma \colon X \longrightarrow Y$ be the blow up of Y in a point. Let $q(x_1, \ldots, x_n)$ be the cubic polynomial which describes the cup form of Y w. r. t. the basis $(\kappa_1, \ldots, \kappa_n)$ of $H^2(Y, \mathbf{Z})$. If $h_0 \in H^2(X, \mathbf{Z})$ is the cohomology class of the exceptional divisor, then $(h_0, \sigma^*\kappa_1, \ldots, \sigma^*\kappa_n)$ is a basis of $H^2(Y, \mathbf{Z})$ w. r. t. which the cup form of X is given by*

$$x_0^3 + q(x_1, \ldots, x_n).$$

ii) *Let $C \subset Y$ be a smooth curve, and $\sigma\colon X \longrightarrow Y$ be the blow up of Y along this curve. Using the same notation as in* i), *the cup form of X is described by the polynomial*

$$q(x_1, \dots, x_n) - 3 \cdot \Big(\sum_{i=1}^{n} (C \,.\, \kappa_i) x_i x_0^2\Big) - \deg_C(N_{C/Y}) x_0^3 \,.$$

Here, $C \,.\, \kappa_i$ stands for the evaluation of the homology class of C on κ_i, and $N_{C/Y}$ is the normal bundle of C in Y.

Proof. This follows easily from [GH], p.602ff. □

1.3. *Complete Intersections in Products of Projective Spaces.*

Let $\mathbf{P}_{n_1} \times \cdots \times \mathbf{P}_{n_r}$ be a product of projective spaces. Write $\mathcal{O}(a_1, \dots, a_r)$ for the invertible sheaf $\pi_1^* \mathcal{O}_{\mathbf{P}_{n_1}}(a_1) \otimes \cdots \otimes \pi_r^* \mathcal{O}_{\mathbf{P}_{n_r}}(a_r)$. Here, π_i is the projection onto the i-th factor. If all the a_i's are positive, this sheaf is very ample. A section in it is given by a multihomogeneous polynomial of multidegree $(a_1, \dots, a_r)$. We denote by

$$\begin{bmatrix} \mathbf{P}_{n_1} & | & a_1^1 & \dots & a_1^m \\ \vdots & & \vdots & & \vdots \\ \mathbf{P}_{n_r} & | & a_r^1 & \dots & a_r^m \end{bmatrix}$$

the family of zero sets of sections of the sheaf

$$\mathcal{O}(a_1^1, \dots, a_r^1) \oplus \cdots \oplus \mathcal{O}(a_1^m, \dots, a_r^m) \,.$$

The members of this family are complete intersections of m hypersurfaces. An iterated application of Theorem 5 shows that a general member X of such a family is smooth and simply connected and that $(h_1, \dots, h_m)$ with $h_i := \pi_i^*(c_1(\mathcal{O}_{\mathbf{P}_{n_i}}(1)))$ is a basis for $H^2(X, \mathbf{Z})$.

2. A Projective Threefold with a Nodal Cubic as Cup Form

Let Y be a smooth member of the family $\begin{bmatrix} \mathbf{P}_4 & | & 1 & 2 \\ \mathbf{P}_1 & | & 1 & 1 \end{bmatrix}$. We first compute the cup form of Y. Let $(\widetilde{h}_1, \widetilde{h}_2)$ be the canonical basis of $H^2(\mathbf{P}_4 \times \mathbf{P}_1, \mathbf{Z})$, and (h_1, h_2) be the basis of $H^2(X, \mathbf{Z})$ as described in 1.3. We compute, e.g.,

$$h_1^2 h_2 = \widetilde{h}_1^2 \widetilde{h}_2 (\widetilde{h}_1 + \widetilde{h}_2)(2\widetilde{h}_1 + \widetilde{h}_2) = 2\widetilde{h}_1^4 \widetilde{h}_2 = 2 \,.$$

Here we have written the cup product followed by evaluation on the fundamental class as multiplication. The cup form of Y is given by the polynomial

$$3x_1^3 + 6x_1^2 x_2 \,.$$