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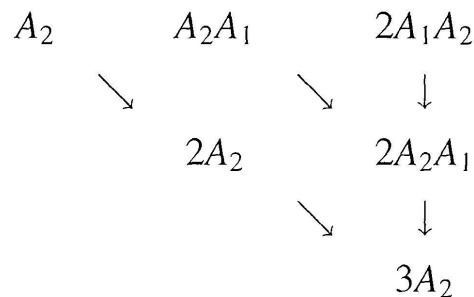
## 2. THE INVARIANT THEORY OF QUATERNARY CUBIC FORMS

2.1. *Stable, Semistable and Nullforms.* The stable and semistable quaternary cubic forms and the quaternary cubic nullforms were determined by Hilbert [Hi] (for the definition of semistable and stable see [Ne], *nullform* means non-semistable form):

THEOREM 3. i) A quaternary cubic form  $f$  is stable (resp. semistable) if and only if the surface  $\{f = 0\}$  has at most singularities of type  $A_1$  (resp.  $A_2$ ).

ii) A quaternary cubic form  $f$  is a nullform if and only if the surface  $\{f = 0\}$  has isolated singularities of type  $A_k$  ( $k \geq 3$ ),  $D_4$ ,  $D_5$ ,  $E_6$ , or  $\tilde{E}_6$ , or if it has non-isolated singularities.

2.2. *Degenerations of Orbits of Semistable Forms.* First, one observes that the semistable forms with closed orbit are precisely the forms whose associated cubic surfaces have three  $A_2$ -singularities. Applying Luna's slice theorem, one then computes the following table of degenerations where we characterize a form by the configuration of singularities on the corresponding cubic surface:



The details can be found in [Sch1], 58ff.

2.3. *The Ring of Invariants.* Proofs of the following results can be found in the paper [Be]. We want to describe the ring  $A := \mathbf{C}[S^3(\mathbf{C}^{4^\vee})]^{SL_4(\mathbf{C})}$ . This is the coordinate ring of the categorical quotient  $S^3(\mathbf{C}^{4^\vee}) // SL_4(\mathbf{C})$ . It is the ring of polynomial expressions in the coefficients of cubic polynomials which are constant on all  $SL_4(\mathbf{C})$ -orbits. In order to describe the ring  $A$ , we first introduce the following vector space

$$S := \left\{ r_1x_1^3 + r_2x_2^3 + r_3x_3^3 + r_4x_4^3 + r_5x_5^3 \mid \sum x_i = 0 \right\}.$$

On  $S$ , there is a natural action of the alternating group  $\mathfrak{A}_5$ , and  $A \subset \mathbf{C}[S]^{\mathfrak{A}_5}$ . This inclusion is constructed as follows: The group of automorphisms  $H$  of the Sylvester pentrahedron naturally acts on  $S$ , and it can be shown that the natural

morphism  $S//H \rightarrow S^3(\mathbf{C}^{4^\vee})//\mathrm{SL}_4(\mathbf{C})$  is birational. This induces the inclusion  $A \subset \mathbf{C}[S]^H$ . Now,  $H$  is a finite group of order 480 obviously containing  $\mathfrak{A}_5$ . Denote by  $\sigma_i$ ,  $i = 1, 2, 3, 4, 5$ , and  $v$  the  $i$ -th symmetric function and the Vandermonde determinant in the  $r_i$ . Then  $\mathbf{C}[S]^{\mathfrak{A}_5} = \mathbf{C}[\sigma_1, \dots, \sigma_5, v]$ .

**THEOREM 4.** *The ring of invariants  $A$  is the subring of  $\mathbf{C}[S]^{\mathfrak{A}_5}$  generated by the following invariant polynomials*

$$I_8 := \sigma_4^2 - 4\sigma_3\sigma_5, \quad I_{16} := \sigma_5^3\sigma_1, \quad I_{24} := \sigma_5^4\sigma_4, \\ I_{32} := \sigma_5^6\sigma_2, \quad I_{40} := \sigma_5^8, \quad I_{100} := \sigma_5^{18}v,$$

which satisfy a relation

$$I_{100}^2 = P(I_8, I_{16}, I_{24}, I_{32}, I_{40}).$$

**2.4. The Discriminant.** Using techniques from the paper [BC], one obtains the following

**PROPOSITION 3.** *The discriminant of quaternary cubic forms is given by the formula*

$$\Delta = (I_8^2 - 64I_{16})^2 - 2^{11}(I_8I_{24} + 8I_{32}).$$

**2.5. Moduli Spaces of Cubic Surfaces.** Define  $\overline{\mathcal{M}}$  to be the hypersurface  $\{I_{100}^2 - P(I_8, I_{16}, I_{24}, I_{32}, I_{40}) = 0\}$  in the weighted projective space  $\mathbf{P}(8, 16, 24, 32, 40) = \mathbf{P}(1, 2, 3, 4, 5)$ . Then  $\mathcal{M} := \overline{\mathcal{M}} \setminus \{\Delta = 0\}$  is a moduli space for non-singular cubic surfaces. On the other hand, every non-singular cubic surface can be obtained as the blow up of  $\mathbf{P}_2$  in six points in general position. The sextuples of points in general position form an open subset  $\mathcal{U} \subset S^6\mathbf{P}_2$  of the sixth symmetric power of  $\mathbf{P}_2$ . Furthermore, there is an action of  $\mathrm{PGL}_3(\mathbf{C})$  on  $\mathcal{U}$ , and the geometric quotient  $\mathcal{N} := \mathcal{U}//\mathrm{PGL}_3(\mathbf{C})$  does exist [Is]. By [Is], §6,  $\mathcal{N}$  is a coarse moduli space for pairs  $(X, L)$  consisting of a cubic surface  $X$  and a globally generated line bundle  $L$  which defines a blow down  $X \rightarrow \mathbf{P}_2$ . Forgetting the line bundle  $L$  provides us with a morphism  $\mathcal{N} \rightarrow \mathcal{M}$ , so that there is a surjection  $f: \mathcal{U} \rightarrow \mathcal{M}$ . Hence, we can view the invariants of quaternary cubic forms as regular functions on  $\mathcal{U}$ . This relates the geometry of the cubic surface to the set of six points. One obtains, e.g.,

PROPOSITION 4. *The set of sextuples in  $\mathcal{U}$  whose associated cubic surface is given by an equation which is not a (nondegenerate) Sylvestrian pentahedral form is the Zariski-closed subset  $\{f^*I_{40} = 0\}$ .*

Of course, a better understanding of the geometric meaning of the other invariants should allow to extend this result.

## II. CUBIC FORMS OF PROJECTIVE THREEFOLDS

### 1. PRELIMINARIES

For the convenience of the reader, we have collected the crucial theorems which we will use in the construction of our examples.

1.1. *The Lefschetz Theorem on Hyperplane Sections.* We summarize Bertini's Theorem and Lefschetz' Theorem in:

THEOREM 5. *Let  $Y$  be a projective manifold,  $L$  a very ample line bundle on  $Y$ , and  $X := Z(s)$  the zero-set of a general section  $s \in H^0(X, L)$ . Then  $X$  is a manifold (connected if  $\dim Y \geq 2$ ), and the inclusion  $\iota: X \hookrightarrow Y$  induces isomorphisms*

$$\begin{aligned} \iota^*: H^i(Y, \mathbf{Z}) &\longrightarrow H^i(X, \mathbf{Z}), & i = 1, \dots, \dim Y - 2; \\ \iota_*: \pi_i(X) &\longrightarrow \pi_i(Y), & i = 1, \dots, \dim Y - 2. \end{aligned}$$

*Proof.* [La], Th. 3.6.7 & Th. 8.1.1.  $\square$

1.2. *Formulas for Blow Ups.* A very simple way to obtain a new manifold from a given one is the blow up in a point or along a smooth curve. The cup form behaves as follows (we will suppose for simplicity that  $H^2(Y, \mathbf{Z})$  is without torsion):

THEOREM 6. i) *Let  $\sigma: X \longrightarrow Y$  be the blow up of  $Y$  in a point. Let  $q(x_1, \dots, x_n)$  be the cubic polynomial which describes the cup form of  $Y$  w. r. t. the basis  $(\kappa_1, \dots, \kappa_n)$  of  $H^2(Y, \mathbf{Z})$ . If  $h_0 \in H^2(X, \mathbf{Z})$  is the cohomology class of the exceptional divisor, then  $(h_0, \sigma^*\kappa_1, \dots, \sigma^*\kappa_n)$  is a basis of  $H^2(Y, \mathbf{Z})$  w. r. t. which the cup form of  $X$  is given by*

$$x_0^3 + q(x_1, \dots, x_n).$$