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Autor: SCHMITT, Alexander

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QUATERNARY CUBIC FORMS AND PROJECTIVE ALGEBRAIC THREEFOLDS

by Alexander SCHMITT

INTRODUCTION

As algebraic geometers, we are interested in a special kind of complex manifolds, namely in complex submanifolds of projective spaces. A submanifold X of $\mathbf{P}_n$ is given as the common zero locus of a set of homogeneous polynomials such that the Jacobi matrix of these polynomials has rank $n - \dim X$ at every point of X. We call such a manifold a *projective algebraic manifold.* The main goal is the classification of projective algebraic manifolds up to biholomorphic equivalence. Now, a projective algebraic manifold is in particular an oriented and closed topological manifold. Moreover, biholomorphic maps are orientation preserving homeomorphisms.

Thus, we obtain a natural approach to the classification of projective algebraic manifolds which can be stated for complex dimension 3 as follows:

Given a six-dimensional, closed, and oriented topological manifold X, describe all projective algebraic threefolds (up to biholomorphic equivalence) whose underlying topological manifold is orientation preservingly homeomorphic to X.

Of course, one does not have a general classification of the respective topological manifolds. However, if we restrict our attention to simply connected, six-dimensional, closed, and oriented topological manifolds with torsion free homology, there is a classification result in the sense of algebraic topology, due to C.T.C. Wall [Wa] and P.E. Jupp [Ju]. This means the classification of simply connected, six-dimensional, closed, and oriented topological manifolds with torsion free homology up to orientation preserving homeomorphy can be reduced to the classification of certain algebraic data, so called admissible systems of invariants.

The explicit classification of these algebraic data can be carried out in the case the second Betti number b_2 is 1 [OV]. But already for $b_2 = 2$, the picture is rather complicated and not yet complete [Sch3]. So, it seems to be a rather hopeless task to classify systems of invariants for $b_2 > 2$. Thus, we restrict ourselves to the consideration of the most important part of the system of invariants of the simply connected, six-dimensional, closed, and oriented topological manifold X, the cup form

$$\begin{array}{cccc} \varphi_X: & S^3H^2(X,\mathbf{Z}) & \longrightarrow & \mathbf{Z} \\ & [a\otimes b\otimes c] & \longmapsto & (a\cup b\cup c)[X]\,. \end{array}$$

Here, $[X]$ is the fundamental class of X. We remark that the assumptions we make on the manifold X imply that the whole cohomology ring of X is determined by φ_X and the third Betti number $b_3(X)$.

We can also replace $\mathbf{Z}$ by $\mathbf{R}$ or $\mathbf{C}$ to obtain a weaker invariant. By our hypothesis, $H^2(X,\mathbf{Z})$ is a free $\mathbf{Z}$-module, and $H^2(X,R) = H^2(X,\mathbf{Z})\otimes_{\mathbf{Z}} R$, $R = \mathbf{R},\mathbf{C}$. If we fix a basis for $H^2(X,R)$, we can identify φ_X with a homogeneous cubic polynomial. On the module of all homogeneous cubic polynomials in b variables, there is an action of $\mathrm{GL}_b(R)$ by substitution of variables. Hence, we obtain a coarse picture of the classification of simply connected, six-dimensional, closed, and oriented topological manifolds with $b_2 = b$ if we determine the normal forms for cubic polynomials over $\mathbf{Z}$ in b variables w. r. t. the action of $\mathrm{GL}_b(\mathbf{Z})$ and if we describe the set of forms φ_X, X being a topological manifold.

For the latter part, we remark that there is a simple criterion to check whether a given cubic polynomial over $\mathbf{Z}$ is of the form φ_X or not (see [Sch2], Cor. 1). For example, this criterion is fulfilled if all coefficients are divisible by 6. The determination of normal forms is again very difficult. However, if we work over the field of complex numbers instead, results are known for up to $b = 4$ variables. The results for $b \leq 3$ variables are easily accessible. On the other hand, the results for $b = 4$ are scattered in the literature of over 100 years. Hence, we have written an extensive summary of the theory of complex quaternary cubic forms. Being interested in (Cubic forms over $\mathbf{Z}$)$/\,\mathrm{GL}_b(\mathbf{Z})$, it is more reasonable to consider the action of $\widetilde{\mathrm{SL}}_b(\mathbf{C}) := \{m \in \mathrm{GL}_b(\mathbf{C}) \mid \det(m) = \pm 1\}$. To simplify things we will consider the action of $\mathrm{SL}_b(\mathbf{C})$ instead. This is the content of Part I.

In the second part, we treat the following weakened form of our original problem:

Which quaternary cubic forms can occur as cup forms of simply connected projective threefolds?

For the case $b \leq 3$, we refer the reader to [OV]. In this part, we have collected a number of examples. We also show that there is a simply connected projective threefold with $b_2 = 3$ whose cup form defines a plane cubic with a node, a problem which remained unsolved in [OV]. We conclude our notes by a brief summary of the author's results concerning the non-realizability of certain *real* cubic polynomials as cup forms of projective threefolds.

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I. Quaternary Cubic Forms

In this section, we will be concerned with the space $S^3(\mathbf{C}^{4^\vee})$ of quaternary cubic forms on which $\mathrm{SL}_4(\mathbf{C})$ acts by substitution of variables. In particular, we will treat the following problems:

1) Find "good" representatives for the orbits in $S^3(\mathbf{C}^{4^\vee})$;

2) Describe the categorical quotient $S^3(\mathbf{C}^{4^\vee})/\!/\,\mathrm{SL}_4(\mathbf{C})$.

(The categorical quotient is an affine algebraic variety whose set of points is in natural bijection with the closed orbits in $S^3(\mathbf{C}^{4^\vee})$. A good introduction to this kind of constructions can be found in [Ne].)