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To the field of indicatrices there corresponds a (Lagrangian) function L on the tangent bundle TM: this function is homogeneous of degree 1 in tangent vectors, and $L^{-1}(1) \cap T_x M = I_x$ for all $x \in M$. This function gives the length of a tangent vector in Finsler geometry. Trajectories of light in Finsler geometry are the extremals of the functional $\int L(q, \dot{q})\, dt$.

THEOREM 1.3. *These extremals are the projections to M of the trajectories of the vector field ξ.*

Thus the Hamiltonian vector field ξ of the Hamiltonian function H describes the propagation of light in an inhomogeneous anisotropic medium. In the case of Minkowski geometry H depends on the momenta variables only. The trajectories of light in Minkowski geometry are straight lines, and the indicatrix is identified with the time-1 front of the origin. The cooriented contact elements of this front are the time-1 images in the geodesic flow of all contact elements at the origin.

Let $N \subset \mathbf{R}^n$ be a cooriented hypersurface in Minkowski space. The geodesic flow trajectories of the foot points of the cooriented contact elements of N will be called (Minkowski) *normals* of N. Note that the normals may change if the coorientation of N is reversed. The reader interested in differential geometry of Finsler manifolds is referred to [Ru], and to [Bu] for the case of Minkowski geometry.

2. MINKOWSKI GEOMETRY ASSOCIATED WITH A PARAMETRIZED CURVE

Return to the situation of the Introduction: $\gamma(t)$ is a smooth closed strictly convex parametrized plane curve satisfying the condition $\left[\gamma''(t), \gamma'''(t)\right] \neq 0$ for all t. The lines $l(t)$ generated by the acceleration vectors $\gamma''(t)$ constitute a smooth transverse line field along $\gamma(t)$. The condition $\left[\gamma''(t), \gamma'''(t)\right] \neq 0$ ensures that infinitesimally close lines from the family $l(t)$ intersect, therefore their envelope is bounded.

Give γ the inward coorientation. Then γ determines a curve $\tilde{\gamma}$ in the space of cooriented contact elements of the plane. The curve $\tilde{\gamma}$ is Legendrian, that is, tangent to the contact structure in the space of cooriented contact elements.

THEOREM 2.1. *There exists a unique, up to a multiplicative constant, Minkowski metric in the plane such that the lines $l(t)$ are the Minkowski normals of the cooriented curve γ.*

Proof. Identify the tangent planes at different points with $\mathbf{R}^2$ by parallel translations. Consider the curve $S(t) = \gamma'(t) \subset \mathbf{R}^2$. Since γ is strictly convex, S is star-shaped. Moreover, $S'(t) = \gamma''(t)$, therefore $[S'(t), S''(t)] \neq 0$ for all t. Thus the curve S is strictly convex.

Assume that the curve $\gamma(t)$ is oriented counterclockwise. Identify the tangent and cotangent planes by the bilinear form $[\,,\,]$: a vector v is considered as the covector $[v,\ ]$. Then one may consider S as a curve in the dual plane $(\mathbf{R}^2)^*$.

Let H be the homogeneous of degree 1 function in $(\mathbf{R}^2)^*$ whose level curve $H^{-1}(1)$ is S. Consider H as a function on $T^*\mathbf{R}^2$ depending on the momenta only, and let ξ be its Hamiltonian vector field. We claim that the cooriented contact element of the curve γ at point $\gamma(t)$ is translated by the field ξ in the direction of the line $l(t)$. The desired Minkowski metric is determined then by the Hamiltonian function H, as explained in the previous section.

To start with, the trajectories of ξ project to straight lines; in local coordinates, $\xi = H_p\ \partial/\partial q$. Since $H(\gamma'(t)) = 1$, one has $dH(\gamma''(t)) = 0$ (here γ' and γ'' are considered as vectors in $(\mathbf{R}^2)^*$). The differential $dH \in ((\mathbf{R}^2)^*)^* = (\mathbf{R}^2)$, and, as a vector in $(\mathbf{R}^2)$, this is $H_p\ \partial/\partial q$. In view of the chosen identification of $(\mathbf{R}^2)^*$ with $(\mathbf{R}^2)$, the equality $dH(\gamma'') = 0$ reads $[H_p\ \partial/\partial q, \gamma''] = 0$ in $(\mathbf{R}^2)$.

Thus $H_p\ \partial/\partial q$ is colinear with γ'' at every point of the curve.

Conversely, the same argument shows that if the trajectories of the Hamiltonian field ξ project to the lines $l(t)$ then H is constant on the curve $S(t) = \gamma'(t)$. This, along with the homogeneity, determines H, and therefore the Minkowski metric, up to a multiplicative constant.

REMARKS. 1) If the function $[\gamma''(t), \gamma'''(t)]$ has zeroes the above theorem still provides a star-shaped figuratrix S which, however, fails to be convex and therefore does not determine a Minkowski metric.

2) Suppose a smooth strictly convex closed nonparametrized plane curve γ is given. Then a choice of a homogeneous function H in $(\mathbf{R}^2)^*$, such that $S = H^{-1}(1)$ is star-shaped, determines a parameterization $\gamma(t)$ with the property that the trajectories of the Hamiltonian vector field ξ project to the lines generated by the vectors $\gamma''(t)$ along γ. Considering S as a curve in $\mathbf{R}^2$, this parameterization is defined by the requirement: $\gamma'(t) \in S$ for all t.

To describe the indicatrix of the Minkowski geometry constructed in the above theorem one needs the following lemma. As before, we identify the tangent and the cotangent planes by the bilinear form [,].

LEMMA 2.2. *Let $S(t)$ be a parametrized strictly convex star-shaped curve in* $(\mathbf{R}^2)^*$. *Then the dual curve is* $S^*(t) = S'(t)/\left[S(t), S'(t)\right]$.

Proof. By definition, the dual curve consists of the vectors $S^*(t) \in \mathbf{R}^2$ such that $< S(t), S^*(t) >= 1$ and $< S'(t), S^*(t) >= 0$. Clearly, the curve $S^*(t) = S'(t)/\left[S(t), S'(t)\right]$ satisfies both equalities, and Lemma 2.2 follows.

This lemma, applied to the curve $S(t) = \gamma'(t)$, along with Theorem 1.2, implies the formula for the indicatrix:

$$I(t) = \gamma''(t)/\left[\gamma'(t), \gamma''(t)\right].$$

This formula gives the plane projection of the velocity vector of the cooriented contact element of the curve γ at point $\gamma(t)$ in the geodesic flow. Notice that the original parameterization $\gamma(t)$ is not, in general, by arc-length in the constructed Minkowski geometry.

Next, we give some explicit formulas in Euclidean terms. Let $\alpha(t)$ be the angle made by the tangent vector $\gamma(t)$ with a fixed direction. Set $f\left(\alpha(t)\right) = \log|\gamma'(t)|$; then

$$\gamma'(t) = e^{f(\alpha(t))}\left(\cos\alpha(t),\ \sin\alpha(t)\right).$$

The plane projection of the vector of the geodesic flow at point $\gamma(t)$ is given by the formula

$$e^{-f(\alpha(t))}\left(f'\left(\alpha(t)\right)\cos\alpha(t) - \sin\alpha(t),\ f'\left(\alpha(t)\right)\sin\alpha(t) + \cos\alpha(t)\right),$$

where prime means $d/d\alpha$.

The function $f(\alpha)$ determines the Minkowski metric. The convexity condition for the indicatrix reads: $1 + (f')^2 > f''$. If $\gamma(t)$ is arc-length parametrized then $f(\alpha) = 1$ identically, and the Minkowski metric is the Euclidean one.