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1. A FINITE ALPHABET

Although g is, compared with the trivially square-free sequence $(k)_{k\in\mathbf{N}}$, economic in the sense that it uses smaller numbers for any finite part, it is unsatisfactory to depend on an *infinite* alphabet. Instead of considering d_μ in the Olive sequence, we now focus on (i_μ, j_μ), i.e. disregarding which disc is involved, we concentrate on the ways the discs are moving. Of these there are only six, namely

$$\alpha := (0,1),\ \ \beta := (1,2),\ \ \gamma := (2,0),\ \ \overline{\alpha} := (1,0),\ \ \overline{\beta} := (2,1),\ \ \overline{\gamma} := (0,2),$$

which will form the alphabet $\mathcal{A} := \{\alpha,\beta,\gamma,\overline{\alpha},\overline{\beta},\overline{\gamma}\}$. J.-P. Allouche *et al.* [2, Theorem 9] have shown that the sequence $c := (i_\mu, j_\mu)_{\mu\in\mathbf{N}}$ (named for N. Claus de Siam, who described the recursive solution in [7]) is square-free by recourse to the language of iterated morphisms. (For another interesting property of this sequence see Allouche and F. Dress [3].) We give a direct proof now, using only the following property of the TH itself:

LEMMA. *If* $\mu = 2^r(2k+1), r,k \in \mathbf{N}_0$, *then*

$$i_\mu = \{(1 + r \bmod 2)\, k\} \mod 3,$$
$$j_\mu = \{(1 + r \bmod 2)\,(k+1)\} \mod 3. \quad \square$$

Proof. a) Let $n \in \mathbf{N}$ be such that $\mu < 2^n$ and put $i = 0, j = 2 - n \bmod 2$ in [13, Proposition 1]. Then, using $d_\mu = r+1$, we get

$$\begin{aligned} i_\mu &= \{k(2 - n \bmod 2)\big((n-r-1) \bmod 2 + 1\big)\} \mod 3 \\ &= \{(1 + r \bmod 2)\, k\} \mod 3, \end{aligned}$$

and similarly for j_μ.

b) As an alternative, we can prove this lemma directly by induction. Assume it is true for $1 \le \mu < 2^n$, $n \in \mathbf{N}_0$, when n discs move from peg 0 to peg $2 - n \bmod 2$. Then $\mu = 2^n$ is the move of disc $n+1$ from 0 to $1 + n \bmod 2$. For $2^n < \mu < 2^{n+1}$, discs 1 to n are transferred from $2 - n \bmod 2$ to $1 + n \bmod 2$; hence move μ is the same as move $\mu - 2^n$ with $0,1,2$ changed to $1,2,0$, respectively, if n is odd, and to $2,0,1$, if n is even. But then, since $\mu - 2^n$ is divisible by the same power of 2 as μ itself, we have $\mu - 2^n = 2^r\left(2(k - 2^{n-r-1}) + 1\right)$, and the formulas follow from $\left((1 + r \bmod 2)2^{n-r-1}\right) \bmod 3 = 2 - n \bmod 2$. $\square$

There are a couple of immediate consequences which we will need later:

COROLLARY.
o) $c_\mu \in \{\alpha, \beta, \gamma\} \Leftrightarrow r \bmod 2 = 0$;
i) $c_\mu \in \left(\{0,1,2\} \setminus \{(2\mu) \bmod 3\}\right)^2$;
ii) $c_\mu = \alpha, \beta, \gamma, \overline{\alpha}, \overline{\beta}, \overline{\gamma} \Leftrightarrow c_{2\mu} = \overline{\gamma}, \overline{\beta}, \overline{\alpha}, \gamma, \beta, \alpha$, *respectively.* □

Proof. (o) is trivial; (i) and (ii) follow from

$$2\mu = (1 + r \bmod 2)\,(k+2),$$

$$1 + (r+1) \bmod 2 = 2(1 + r \bmod 2)\,,$$

both taken modulo 3, respectively. □

REMARK. Another direct consequence of the Lemma is (cf. [3, p. 10]):

$$c_\mu = \alpha, \beta, \gamma, \overline{\alpha}, \overline{\beta}, \overline{\gamma} \Leftrightarrow$$
$$\exists\, s, l \in \mathbf{N}_0 : \frac{\mu}{4^s} = 6l+1, 6l+3, 6l+5, 12l+10, 12l+6, 12l+2,$$

respectively.

Our asymmetric choice of the first move being from 0 to 1 is here reflected in having, in some sense, twice as many unbarred as barred symbols in c, as remarked in [3, p.13]. □

Now we can prove the result of Allouche *et al.*:

THEOREM 1. *c is square-free.* □

Proof. Assume

$$\exists\, m \in \mathbf{N}_0\, \exists\, l \in \mathbf{N} \quad \forall \nu \in \{m+1, \dots, m+l\} : c_{\nu+l} = c_\nu\,.$$

If l is odd, then ν and $\nu + l$ have different parity. So every $\nu \in \{m+1, \dots, m+2l\}$ has an even number of factors 2 by Corollary (o). Since of four consecutive numbers one has exactly one factor $2, l$ can only be 1. This, however, contradicts Corollary (i). Hence l must be even. But then, by virtue of Corollary (ii), the same argument as in the proof of Theorem 0 applies. □