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$$\theta(p^{-1}q) = \theta(C_n^m)$$

$$\text{and } \theta((p^{-1}q)^{n+2}) = \theta((C_n^m)^{n+2}) = \theta((C_n^{n+2})^m) \stackrel{5.6.v)}{=} \theta(1) = 1.$$

By Lemma 5.4, there is a homomorphism  $\alpha: F \rightarrow T_1/N$  defined on generators by  $\alpha(A) = \theta(A)$  and  $\alpha(B) = \theta(B)$ . If  $p^{-1}q \neq 1$ , then  $(p^{-1}q)^{n+2} \neq 1$ , and so  $\alpha(F)$  is a proper quotient group of  $F$ . Since every proper quotient group of  $F$  is Abelian by Theorem 4.3,  $\theta(AB) = \theta(BA)$ . If  $p^{-1}q = 1$ , then  $m, n > 0$  and  $1 = \theta(C_n^m) \stackrel{5.6.iv)}{=} \theta(C_{n+1}^{m+1})\theta(X_{m-1}^{-1})$  and hence  $\theta(X_{m-1}^{n+3}) = \theta((C_{n+1}^{m+1})^{n+3}) = \theta((C_{n+1}^{n+3})^{m+1}) = \theta(1) = 1$ . It follows as before that  $\theta(AB) = \theta(BA)$ . Hence  $\theta(A^{-1}BA) = \theta(B)$ , so  $\theta(A^{-1}C) = \theta(BA^{-2}C)$  by relation 4). Hence  $\theta(BA^{-1}) = 1$ , and so  $\theta(B) = 1$  by relation 3). This implies that  $\theta(A) = 1$ . It now follows from relation 5) that  $\theta(C) = 1$ . Thus  $N = T_1$ , and so  $T_1$  is simple.  $\square$

COROLLARY 5.9.  $T_1$  is isomorphic to  $T$ .

## §6. THOMPSON'S GROUP $V$

As with the previous section, the material in this section is mainly from unpublished notes of Thompson [T1]; [T1] contains the statements of the lemmas (except for Lemma 6.2) and the statement and proof of Theorem 6.9, but does not contain the proofs of the lemmas.

Let  $V$  be the group of right-continuous bijections of  $S^1$  that map images of dyadic rational numbers to images of dyadic rational numbers, that are differentiable except at finitely many images of dyadic rational numbers, and such that, on each maximal interval on which the function is differentiable, the function is linear with derivative a power of 2. As before, it is easy to prove that  $V$  is a group.

We can associate tree diagrams with elements of  $V$  as we did for  $F$  and  $T$ , except that now we need to label the leaves of the domain and range trees to indicate the correspondence between the leaves. For example, reduced tree diagrams for  $A$ ,  $B$ , and  $C$  are given in Figure 16.

Using the identification of  $S^1$  as the quotient of  $[0, 1]$ , define  $\pi_0: S^1 \rightarrow S^1$  by

$$\pi_0(x) = \begin{cases} \frac{x}{2} + \frac{1}{2}, & 0 \leq x < \frac{1}{2} \\ 2x - 1, & \frac{1}{2} \leq x < \frac{3}{4} \\ x, & \frac{3}{4} \leq x < 1. \end{cases}$$

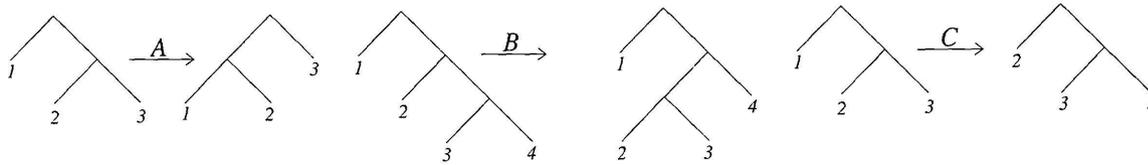


FIGURE 16

Reduced tree diagrams for  $A$ ,  $B$ , and  $C$

We define elements  $X_n$  and  $C_n$  of  $V$  as before. That is,  $X_0 = A$ ,  $X_n = A^{-n+1}BA^{n-1}$  for an integer  $n \geq 1$ , and  $C_n = A^{-n+1}CB^{n-1}$  for an integer  $n \geq 1$ . Define  $\pi_n$ ,  $n \geq 1$ , by  $\pi_1 = C_2^{-1}\pi_0C_2$  and  $\pi_n = A^{-n+1}\pi_1A^{n-1}$  for  $n \geq 2$ . Reduced tree diagrams from  $\pi_0$ ,  $\pi_1$ ,  $\pi_2$ , and  $\pi_3$  are given in Figure 17.

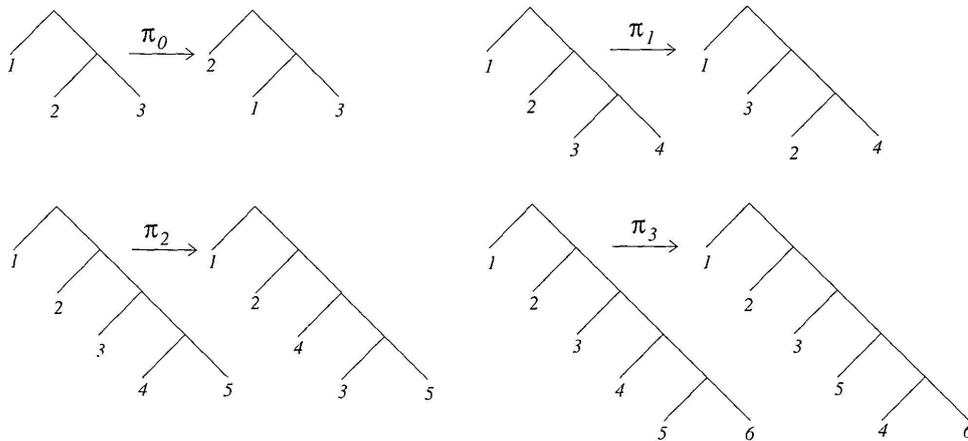


FIGURE 17

Reduced tree diagrams for  $\pi_i$ ,  $0 \leq i \leq 3$

It is easy to see for every positive integer  $n$  that  $\pi_0, \dots, \pi_{n-1}$  generate a subgroup of  $V$  isomorphic with the symmetric group of all permutations of the  $n + 1$  intervals  $[0, 1 - 2^{-1}]$ ,  $[1 - 2^{-1}, 1 - 2^{-2}]$ ,  $[1 - 2^{-2}, 1 - 2^{-3}]$ ,  $\dots$ ,  $[1 - 2^{-n}, 1 - 2^{-(n+1)}]$ . Furthermore  $\pi_0, \dots, \pi_{n-1}$  and  $C_n$  generate a subgroup of  $V$  isomorphic with the symmetric group of all permutations of the  $n + 2$  intervals  $[0, 1 - 2^{-1}]$ ,  $[1 - 2^{-1}, 1 - 2^{-2}]$ ,  $[1 - 2^{-2}, 1 - 2^{-3}]$ ,  $\dots$ ,  $[1 - 2^{-n}, 1 - 2^{-(n+1)}]$ ,  $[1 - 2^{-(n+1)}, 1]$  for every positive integer  $n$ .

LEMMA 6.1. *The elements  $A$ ,  $B$ ,  $C$ , and  $\pi_0$  generate  $V$  and satisfy the following relations :*

- 1)  $[AB^{-1}, X_2] = 1$  ;
- 2)  $[AB^{-1}, X_3] = 1$  ;
- 3)  $C_1 = BC_2$  ;
- 4)  $C_2X_2 = BC_3$  ;

- 5)  $C_1A = C_2^2$ ;
- 6)  $C_1^3 = 1$ ;
- 7)  $\pi_1^2 = 1$ ;
- 8)  $\pi_1\pi_3 = \pi_3\pi_1$ ;
- 9)  $(\pi_2\pi_1)^3 = 1$ ;
- 10)  $X_3\pi_1 = \pi_1X_3$ ;
- 11)  $\pi_1X_2 = B\pi_2\pi_1$ ;
- 12)  $\pi_2B = B\pi_3$ ;
- 13)  $\pi_1C_3 = C_3\pi_2$ ; and
- 14)  $(\pi_1C_2)^3 = 1$ .

*Proof.* Let  $H$  be the subgroup of  $V$  generated by  $A$ ,  $B$ ,  $C$ , and  $\pi_0$ . To prove that  $H = V$ , it suffices to prove that if  $R$  and  $S$  are  $\mathcal{T}$ -trees with  $n$  leaves labeled by  $1, \dots, n$ , then there is an element of  $H$  with domain tree  $R$  and range tree  $S$  which preserves labels. Since  $H$  is a group and  $A$  and  $B$  generate the subgroup  $F$  of  $V$ , we can assume that  $R = S = \mathcal{T}_{n-1}$ . So assume that  $R = S = \mathcal{T}_{n-1}$ . Each element of the subgroup of  $V$  generated by  $\pi_0$  and  $C_{n-2}$  has a tree diagram with domain tree and range tree  $\mathcal{T}_{n-1}$ , and this subgroup is isomorphic to the symmetric group  $\Sigma_n$ , acting on the leaves of  $\mathcal{T}_{n-1}$ . Hence there is an element of  $V$  with domain tree  $R$  and range tree  $S$  which preserves labels, and  $H = V$ .

It follows from Lemma 5.2 that relations 1)-6) are satisfied. Relations 7), 8), 9), 13), and 14) follow easily from the viewpoint of permutations. Relation 10) is true because the supports of  $\pi_1$  and  $X_3$  are disjoint. Relations 11) and 12) can be established by verifying that the reduced tree diagrams for the two elements are the same; the tree diagrams are computed in Figures 18 and 19.  $\square$

The group  $V_1$  will be defined via generators and relators. There will be four generators,  $A$ ,  $B$ ,  $C$ , and  $\pi_0$ . We introduce words  $X_n$ ,  $C_n$ , and  $\pi_n$  as before. That is,  $X_0 = A$ ,  $X_n = A^{-n+1}BA^{n-1}$  for an integer  $n \geq 1$ ,  $C_n = A^{-n+1}CB^{n-1}$  for an integer  $n \geq 1$ ,  $\pi_1 = C_2^{-1}\pi_0C_2$ , and  $\pi_n = A^{-n+1}\pi_1A^{n-1}$  for  $n \geq 2$ .

Let

$$V_1 = \langle A, B, C, \pi_0 : [AB^{-1}, X_2], [AB^{-1}, X_3], BC_2(C_1)^{-1}, BC_3(C_2X_2)^{-1}, \\ C_2^2(C_1A)^{-1}, C_1^3, \pi_1^2, \pi_3\pi_1(\pi_1\pi_3)^{-1}, (\pi_2\pi_1)^3, \pi_1X_3(X_3\pi_1)^{-1}, \\ B\pi_2\pi_1(\pi_1X_2)^{-1}, B\pi_3(\pi_2B)^{-1}, C_3\pi_2(\pi_1C_3)^{-1}, (\pi_1C_2)^3 \rangle$$

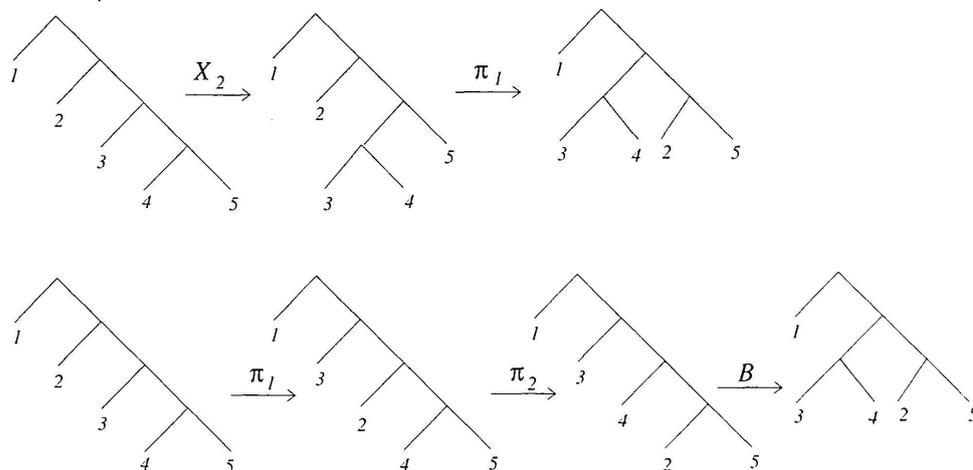


FIGURE 18

Reduced tree diagrams for  $\pi_1 X_2$  and  $B\pi_2\pi_1$

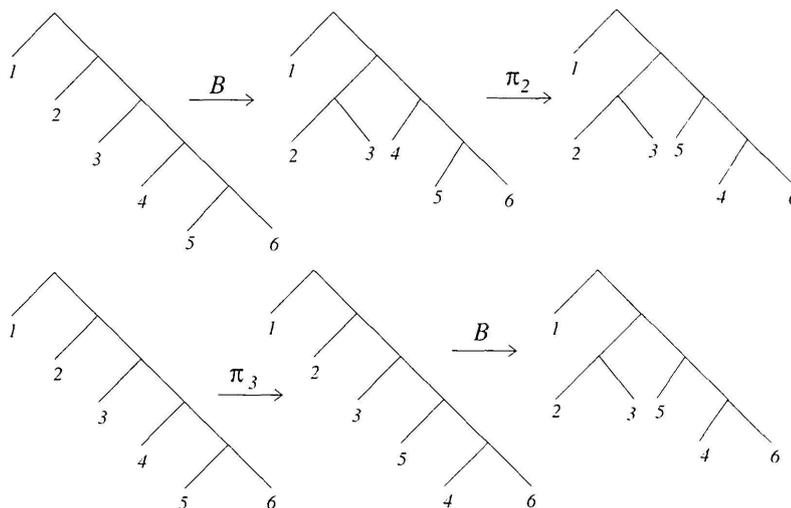


FIGURE 19

Reduced tree diagrams for  $\pi_2 B$  and  $B\pi_3$

We will prove that  $V_1$  is simple. Since there is a surjection from  $V_1$  to  $V$  by Lemma 6.1, it will follow that  $V_1 \cong V$  and  $V$  is simple.

Lemmas 6.3-6.8 contain the relations we need among the  $\pi_i$ 's, the  $X_i$ 's, and the  $C_i$ 's. Lemma 6.2 isolates some parts of them that will be needed in the proof of Lemma 6.3.

LEMMA 6.2. *Let  $i$  be a positive integer and let  $j$  be a integer.*

- i) *If  $0 \leq j < i$ , then  $\pi_i X_j = X_j \pi_{i+1}$ .*
- ii) *If  $j \geq i + 2$ , then  $\pi_i X_j = X_j \pi_i$ .*
- iii) *If  $i > j > 0$ , then  $C_i \pi_j = \pi_{j-1} C_i$ .*

*Proof.* We begin the proof of i) by proving that  $AB^{-1}$  commutes with  $X_n$  and  $\pi_n$  for every integer  $n \geq 2$ . For this let  $H$  be the centralizer of  $AB^{-1}$  in  $V_1$ . Theorem 3.4 easily implies that  $H$  contains  $X_n$  for every integer  $n \geq 2$ . We prove that  $\pi_n \in H$  for every integer  $n \geq 2$  by induction on  $n$ . For  $n = 2$  we have  $\pi_3 = A^{-1}\pi_2A$ , and the relator  $B\pi_3(\pi_2B)^{-1}$  gives  $\pi_3 = B^{-1}\pi_2B$ . Hence  $\pi_2 \in H$ . Now let  $n$  be an integer with  $n \geq 2$ , and suppose that  $\pi_n \in H$ . Since  $H$  contains  $\pi_n$ ,  $X_n$ , and  $X_{n+1}$ ,  $A^{n-1}HA^{-n+1}$  contains  $\pi_1$ ,  $X_1$ , and  $X_2$ . Thus the relator  $B\pi_2\pi_1(\pi_1X_2)^{-1}$  easily gives  $\pi_2 \in A^{n-1}HA^{-n+1}$ , and so  $\pi_{n+1} \in H$ . This proves that  $AB^{-1}$  commutes with  $X_n$  and  $\pi_n$  for every integer  $n \geq 2$ .

We now prove i) by induction on  $j$ . If  $j = 0$ , then i) is clear. Suppose that  $j = 1$  and that  $i$  is an integer with  $i > 1$ . We have  $A^{-1}\pi_iA = \pi_{i+1}$ , and the previous paragraph shows that  $AB^{-1}\pi_iBA^{-1} = \pi_i$ . These identities imply that  $B^{-1}\pi_iB = \pi_{i+1}$ , which gives ii) when  $j = 1$ . Now suppose that  $j > 1$  and that  $i$  is an integer with  $i > j$ . We have  $\pi_{i-j+1}X_1 = X_1\pi_{i-j+2}$ , and so  $A^{-j+1}\pi_{i-j+1}A^{j-1}A^{-j+1}X_1A^{j-1} = A^{-j+1}X_1A^{j-1}A^{-j+1}\pi_{i-j+2}A^{j-1}$ . Hence  $\pi_iX_j = X_j\pi_{i+1}$ . This proves i).

Since  $\pi_1X_3 = X_3\pi_1$ ,  $\pi_2X_4 = A^{-1}\pi_1X_3A = A^{-1}X_3\pi_1A = X_4\pi_2$ .  $B\pi_2\pi_1X_4 = \pi_1X_2X_4 = \pi_1X_3X_2 = X_3\pi_1X_2 = X_3B\pi_2\pi_1 = BX_4\pi_2\pi_1 = B\pi_2X_4\pi_1$ , and so  $\pi_1X_4 = X_4\pi_1$ . If  $n \geq 4$  and  $\pi_1X_n = X_n\pi_1$ , then  $X_3\pi_1X_{n+1} = \pi_1X_3X_{n+1} = \pi_1X_nX_3 = X_n\pi_1X_3 = X_nX_3\pi_1 = X_3X_{n+1}\pi_1$  and so  $\pi_1X_{n+1} = X_{n+1}\pi_1$ . Hence it follows by induction that  $\pi_1X_j = X_j\pi_1$  if  $j \geq 3$ . If  $i, j$  are positive integers and  $j \geq i + 2$ ,  $\pi_iX_j = A^{-i+1}\pi_1A^{i-1}A^{-i+1}X_{j-i+1}A^{i-1} = A^{-i+1}\pi_1X_{j-i+1}A^{i-1} = A^{-i+1}X_{j-i+1}\pi_1A^{i-1} = X_j\pi_i$ . This proves ii).

We prove iii) by induction on  $j$  and  $i$ . We have  $C_3\pi_2 = \pi_1C_3$ . If  $2 < i$  and  $C_i\pi_2 = \pi_1C_i$ , then  $X_iC_{i+1}\pi_2 = C_i\pi_2 = \pi_1C_i = \pi_1X_iC_{i+1} = X_i\pi_1C_{i+1}$  and hence  $C_{i+1}\pi_2 = \pi_1C_{i+1}$ . It follows by induction on  $i$  that  $C_i\pi_2 = \pi_1C_i$  if  $i > 2$ . If  $1 < j < i$  and  $C_i\pi_j = \pi_{j-1}C_i$ , then  $C_{i+1}\pi_{j+1} = C_{i+1}B^{-1}B\pi_{j+1} = C_{i+1}B^{-1}\pi_jB = A^{-1}C_iBB^{-1}\pi_jB = A^{-1}C_i\pi_jB = A^{-1}\pi_{j-1}C_iB = A^{-1}\pi_{j-1}AA^{-1}C_iB = \pi_jC_{i+1}$ . It follows by induction on  $j$  that  $C_i\pi_j = \pi_{j-1}C_j$  if  $1 < j < i$ .

To finish the proof of iii), it remains to show that  $C_i\pi_1 = \pi_0C_i$  if  $1 < i$ . Since  $\pi_1 = C_2^{-1}\pi_0C_2$ ,  $C_2\pi_1 = \pi_0C_2$ . Suppose  $i \geq 2$  and  $C_i\pi_1 = \pi_0C_i$ . Since  $C_iA = C_{i+1}^2$  and  $\pi_1A = A\pi_2$ ,  $C_{i+1}^2\pi_2 = C_iA\pi_2 = C_i\pi_1A = \pi_0C_iA = \pi_0C_{i+1}^2$ . But  $\pi_1C_{i+1} = C_{i+1}\pi_2$ , so  $C_{i+1}\pi_1C_{i+1} = C_{i+1}^2\pi_2 = \pi_0C_{i+1}^2$  and hence  $C_{i+1}\pi_1 = \pi_0C_{i+1}$ . It follows by induction that  $C_i\pi_1 = \pi_0C_i$  if  $1 < i$ .  $\square$

LEMMA 6.3. *If  $i$  is a nonnegative integer, then*

$$i) \pi_i^2 = 1,$$

ii)  $(\pi_{i+1}\pi_i)^3 = 1$ , and

iii)  $\pi_i\pi_j = \pi_j\pi_i$  if  $j \geq i + 2$ .

*Proof.*  $\pi_1^2 = 1$  from the definition of  $V_1$ , and since the  $\pi_i$ 's are conjugate to each other,  $\pi_i^2 = 1$  for  $i \geq 0$ .

$(\pi_2\pi_1)^3 = 1$  is one of the defining relations. Lemma 6.2.iii) shows that  $\pi_{i+1}\pi_i$  is conjugate to  $\pi_2\pi_1$  for every nonnegative integer  $i$ . Hence  $(\pi_{i+1}\pi_i)^3 = 1$  for every nonnegative integer  $i$ . This proves ii).

We may likewise use Lemma 6.2.iii) to reduce the proof of iii) to the case in which  $i = 1$ . Since  $\pi_1\pi_3 = \pi_3\pi_1$ ,  $\pi_2\pi_4 = A^{-1}\pi_1\pi_3A = A^{-1}\pi_3\pi_1A = \pi_4\pi_2$ . Since  $\pi_1\pi_3 = \pi_3\pi_1$ ,  $\pi_1\pi_3X_2 = \pi_3\pi_1X_2$ ,  $\pi_1X_2\pi_4 = \pi_3X_1\pi_2\pi_1$ ,  $X_1\pi_2\pi_1\pi_4 = X_1\pi_4\pi_2\pi_1 = X_1\pi_2\pi_4\pi_1$ , and hence  $\pi_1\pi_4 = \pi_4\pi_1$ . If  $n \geq 4$  and  $\pi_1\pi_n = \pi_n\pi_1$ , then  $X_3\pi_1\pi_{n+1} = \pi_1X_3\pi_{n+1} = \pi_1\pi_nX_3 = \pi_n\pi_1X_3 = \pi_nX_3\pi_1 = X_3\pi_{n-1}\pi_1$ . It follows by induction that  $\pi_1\pi_j = \pi_j\pi_1$  if  $j \geq 3$ . This proves iii).  $\square$

LEMMA 6.4. *If  $i$  and  $j$  are nonnegative integers, then*

i)  $\pi_iX_j = X_j\pi_i$  if  $j \geq i + 2$ ,

ii)  $\pi_iX_{i+1} = X_i\pi_{i+1}\pi_i$ ,

iii)  $\pi_iX_i = X_{i+1}\pi_i\pi_{i+1}$ , and

iv)  $\pi_iX_j = X_j\pi_{i+1}$  if  $0 \leq j < i$ .

*Proof.* If  $i > 0$ , then i) is Lemma 6.2.ii). For  $i = 0$  suppose that  $n$  is an integer with  $j < n$ . Then  $\pi_0X_jC_{n+1} = \pi_0C_nX_{j+1} = C_n\pi_1X_{j+1} = C_nX_{j+1}\pi_1 = X_jC_{n+1}\pi_1 = X_j\pi_0C_{n+1}$  by Lemmas 5.5.ii), 6.2.iii), and 6.2.ii). Hence  $\pi_0X_j = X_j\pi_0$  if  $j \geq 2$ . This proves i).

For ii), the case  $i = 1$  is one of the defining relations. Since  $\pi_1X_2 = B\pi_2\pi_1$ , Lemmas 5.5.ii) and 6.2.iii) give that  $\pi_0BC_3 = \pi_0C_2X_2 = C_2\pi_1X_2 = C_2B\pi_2\pi_1 = AC_3\pi_2\pi_1 = A\pi_1\pi_0C_3$ . This implies that  $\pi_0B = A\pi_1\pi_0$ , which gives ii) when  $i = 0$ . If  $i > 1$ , then conjugating the relation  $\pi_1X_2 = X_1\pi_2\pi_1$  by  $A^{i-1}$  gives  $\pi_iX_{i+1} = X_i\pi_{i+1}\pi_i$ . This proves ii).

iii) follows immediately from ii) since each  $\pi_i$  has order 2.

iv) is Lemma 6.2.i).  $\square$

LEMMA 6.5. *Let  $n$  and  $k$  be positive integers with  $n > k$ . Then*

i)  $C_n\pi_k = \pi_{k-1}C_n$ ,

ii)  $C_n\pi_0 = \pi_0 \cdots \pi_{n-1}C_n^2$ ,

iii)  $C_n^2\pi_0 = \pi_{n-1} \cdots \pi_0C_n$ , and

iv)  $C_n^3\pi_0 = \pi_{n-1}C_n^3$ .

*Proof.* i) is Lemma 6.2.iii).

We prove ii) by induction. Since  $(\pi_1 C_2)^3 = 1$ ,  $(C_2 \pi_1)^3 = 1$ . This implies that  $C_2 \pi_1 C_2 = \pi_1 C_2^{-1} \pi_1$ , and hence that  $C_2^2 \pi_1 C_2^{-1} = C_2 \pi_1 C_2^{-1} \pi_1 C_2^2$  by Lemma 5.6.v). Hence  $C_2 \pi_0 = C_2(C_2 \pi_1 C_2^{-1}) = (C_2 \pi_1 C_2^{-1}) \pi_1 C_2^2 = \pi_0 \pi_1 C_2^2$ , which proves ii) when  $n = 2$ . Suppose that  $n \geq 2$  and  $C_n \pi_0 = \pi_0 \cdots \pi_{n-1} C_n^2$ . Then

$$\begin{aligned} X_n C_{n+1} \pi_0 &= C_n \pi_0 = \pi_0 \cdots \pi_{n-1} C_n^2 = \pi_0 \cdots \pi_{n-1} C_n X_n C_{n+1} \\ &= \pi_0 \cdots \pi_{n-1} X_{n-1} C_{n+1}^2 = \pi_0 \cdots \pi_{n-2} X_n \pi_{n-1} \pi_n C_{n+1}^2 \\ &= X_n \pi_0 \cdots \pi_{n-2} \pi_{n-1} \pi_n C_{n+1}^2, \end{aligned}$$

and hence  $C_{n+1} \pi_0 = \pi_0 \cdots \pi_n C_{n+1}^2$ . ii) now follows by induction.

iii) follows from ii):

$$C_n = (C_n \pi_0) \pi_0 = (\pi_0 \cdots \pi_{n-1} C_n^2) \pi_0,$$

so  $C_n^2 \pi_0 = (\pi_0 \cdots \pi_{n-1})^{-1} C_n = \pi_{n-1} \cdots \pi_0 C_n$ .

iv) follows from i), ii), and iii):

$$\begin{aligned} C_n^3 \pi_0 &= C_n(C_n^2 \pi_0) = C_n(\pi_{n-1} \cdots \pi_0 C_n) = \pi_{n-2} \cdots \pi_0 C_n(\pi_0 C_n) \\ &= \pi_{n-2} \cdots \pi_0 (\pi_0 \cdots \pi_{n-1} C_n^2) C_n = \pi_{n-1} C_n^3. \quad \square \end{aligned}$$

LEMMA 6.6. *Let  $k$ ,  $m$ , and  $n$  be integers with  $0 \leq m < n + 2$  and  $0 \leq k < n$ . Then*

- i) if  $m \leq k$ ,  $C_n^m \pi_k = \pi_{k-m} C_n^m$ ,
- ii) if  $m = k + 1$ ,  $C_n^m \pi_k = \pi_0 \cdots \pi_{n-1} C_n^{m+1}$ ,
- iii) if  $m = k + 2$ ,  $C_n^m \pi_k = \pi_{n-1} \cdots \pi_0 C_n^{m-1}$ , and
- iv) if  $m > k + 2$ ,  $C_n^m \pi_k = \pi_{k+(n+2-m)} C_n^m$ .

*Proof.* i) follows from Lemma 6.5.i) by induction.

Now consider ii). If  $n \geq 2$  and  $m = k + 1$ , then by Lemmas 6.6.i) and 6.5.ii)  $C_n^m \pi_k = C_n C_n^k \pi_k = C_n \pi_0 C_n^k = \pi_0 \cdots \pi_{n-1} C_n^2 C_n^k = \pi_0 \cdots \pi_{n-1} C_n^{m+1}$ , which proves ii) if  $n \geq 2$ . By Lemmas 5.6.i), 6.3.i), and 6.5.iv),  $C^2 B = C_2^3 = \pi_1^2 C_2^3 = \pi_1 C_2^3 \pi_0$ . Hence  $C^2 B \pi_0 = \pi_1 C_2^3 = \pi_0 \pi_0 \pi_1 C_2^3 = \pi_0 C_2 \pi_0 C_2 = \pi_0 C_2^2 \pi_1$  by Lemmas 6.3.i), 6.5.ii), and 6.5.i). Hence  $C^2 B \pi_0 \pi_1 = \pi_0 C_2^2$ , and so  $C^2 \pi_0 A = \pi_0 C A$  by Lemmas 6.4.iii) and 5.5.iii). This gives  $C^2 \pi_0 = \pi_0 C$ , and hence  $C \pi_0 = C(\pi_0 C) C^{-1} = C(C^2 \pi_0) C^{-1} = \pi_0 C^2$ . This completes the proof of ii).

If  $n = 1$ , then the assumptions of iii) imply that  $k = 0$  and  $m = 2$ , and so iii) becomes  $C_1^2\pi_0 = \pi_0C_1$ , hence  $C^2\pi_0 = \pi_0C$ . This was proved in the above paragraph. If  $n \geq 2$  and  $m = k + 2$ , then by Lemmas 6.6.i) and 6.5.iii)  $C_n^m\pi_k = C_n^2C_n^k\pi_k = C_n^2\pi_0C_n^k = \pi_{n-1} \cdots \pi_0C_nC_n^k = \pi_{n-1} \cdots \pi_0C_n^{m-1}$ , which proves iii).

To prove iv), suppose that  $m > k + 2$ . Then by Lemmas 6.6.i) and 6.5.iv)  $C_n^m\pi_k = C_n^{m-k-3}C_n^3C_n^k\pi_k = C_n^{m-k-3}C_n^3\pi_0C_n^k = C_n^{m-k-3}\pi_{n-1}C_n^{k+3} = \pi_{n-1-(m-k-3)}C_n^m$ , which proves iv).  $\square$

For each positive integer  $n$ , let  $\Pi(n)$  be the subgroup of  $V_1$  generated by  $\{\pi_0, \dots, \pi_{n-1}\}$ , and let  $\Pi = \cup_{n \in \mathbf{N}} \Pi(n)$ .

Let  $\Sigma$  be the group of permutations of  $\mathbf{N}$  with finite support. Then

$$\Sigma = \langle s_0, s_1, s_2, \dots : (s_i)^2 \text{ for all } i, \\ (s_i s_{i+1})^3 \text{ for all } i, \\ (s_i s_j)^2 \text{ for all } i \text{ and all } j \geq i + 2 \rangle.$$

Furthermore, in every proper quotient group of  $\Sigma$ , the image of  $s_0$  is the image of  $s_1$ . Since  $\Pi$  is a quotient group of  $\Sigma$  and  $\pi_0 \neq \pi_1$  in  $V$ ,  $\Pi$  is isomorphic to  $\Sigma$ .

Following the terminology for  $F$ , an element of  $V_1$  which is a product of nonnegative powers of the  $X_i$ 's will be called *positive* and an inverse of a positive element will be called *negative*.

LEMMA 6.7. *If  $p$  is a positive element of  $V_1$  and  $\pi \in \Pi$ , then  $\pi p = p' \pi'$  for some positive element  $p'$  and some  $\pi' \in \Pi$ .*

*Proof.* Lemma 6.7 follows from Lemma 6.4.  $\square$

LEMMA 6.8.

i) *If  $m, n$  are positive integers with  $m < n + 2$  and if  $\pi \in \Pi(n)$ , then  $C_n^m\pi = \pi' C_n^{m'}$  for some  $\pi' \in \Pi(n)$  and some positive integer  $m'$  with  $m' < n + 2$ .*

ii) *For each  $n \in \mathbf{N}$ , the subgroup of  $V_1$  generated by  $\Pi(n)$  and  $C_n$  is finite.*

*Proof.* i) follows from Lemmas 6.6 and 5.6.v). ii) follows from i) and Lemma 5.6.v).  $\square$

THEOREM 6.9.  $V_1$  is simple.

*Proof.* Suppose  $N$  is a nontrivial normal subgroup of  $V_1$ , and let  $\theta: V_1 \rightarrow V_1/N$  be the quotient homomorphism. Then there is an element  $g \in V_1$  with  $g \neq 1$  and  $\theta(g) = 1$ . By Lemmas 5.6.iii), 5.6.iv), 6.7, 6.8.i) and Theorem 5.7 we have  $g = p\pi C_n^m q^{-1}$  for some positive elements  $p$  and  $q$ , some integers  $m, n$  with  $0 \leq m < n + 2$ , and some element  $\pi \in \Pi(n)$ . Then  $\theta(\pi C_n^m) = \theta(p^{-1}q)$ . Lemma 6.8.ii) implies that  $\pi C_n^m$  has finite order, say,  $k$ . Furthermore the subgroup of  $V_1$  generated by  $A$  and  $B$  is torsion-free because it maps injectively to  $F \subseteq V$  by Theorem 3.4. Hence either  $(p^{-1}q)^k \neq 1$  and  $\theta((p^{-1}q)^k) = 1$  or  $\pi C_n^m \neq 1$  and  $\theta(\pi C_n^m) = 1$ . Suppose that  $\pi C_n^m \neq 1$  and  $\theta(\pi C_n^m) = 1$ . If  $m = 0$ , then  $\pi \neq 1$  and  $\theta(\pi) = 1$ . This implies that  $\theta(\pi_0) = \theta(\pi_1)$ , and hence by Lemma 6.5 that  $\theta(\pi_0 C_2) = \theta(C_2 \pi_1) = \theta(C_2 \pi_0) = \theta(\pi_0 \pi_1 C_2^2)$ . But then  $\theta(\pi_1 C_2) = 1$ , so we may assume that  $m > 0$ . Next suppose that  $m > 0$ . Then  $\pi C_n^m = \pi X_{n+1-m} C_{n+1}^m$  by Lemma 5.6.iii). Lemma 6.4 implies that there exists a nonnegative integer  $i$  and  $\pi' \in \Pi(n+1)$  such that  $\pi C_n^m = X_i \pi' C_{n+1}^m$ . Thus we are in the above case in which  $(p^{-1}q)^k \neq 1$  and  $\theta((p^{-1}q)^k) = 1$ .

In each case there is an element  $h \in V_1$  such that  $h \neq 1$ ,  $\theta(h) = 1$ , and  $h$  can be represented as a word in  $A^{\pm 1}$ ,  $B^{\pm 1}$ , and  $C^{\pm 1}$ . Let  $\alpha: T_1 \rightarrow V_1/N$  be the homomorphism defined by  $\alpha(A) = \theta(A)$ ,  $\alpha(B) = \theta(B)$ , and  $\alpha(C) = \theta(C)$ . Then there is an element  $h' \in T_1$  with  $h' \neq 1$  and  $\alpha(h') = 1$ . Since  $T_1$  is simple by Theorem 5.8,  $\theta(A) = \theta(B) = \theta(C) = 1$ . Because  $\pi_i$  and  $\pi_j$  are conjugate via a power of  $A$ ,  $\theta(\pi_i) = \theta(\pi_j)$  for all nonnegative integers  $i$  and  $j$ . By Lemma 6.6.ii) with  $k = 1$ ,  $m = 2$  and  $n = 2$ ,  $\theta(\pi_1) = \theta(C_2^2 \pi_1) = \theta(\pi_0 \pi_1 C_2^3) = \theta(\pi_0 \pi_1)$ , and hence  $\theta(\pi_0) = 1$ . This implies that the quotient group is trivial.  $\square$

## §7. PIECEWISE INTEGRAL PROJECTIVE STRUCTURES

The definition of piecewise integral projective structures is due to W. Thurston. These structures arise naturally on the boundaries of Teichmüller spaces of surfaces. The interpretations of  $F$  and  $T$  as groups of piecewise integral projective homeomorphisms are also due to Thurston; we learned this from him in 1975. Greenberg [Gr] used this interpretation in his study of these groups.

Fix a positive integer  $n$ .