§4. Further properties of F

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§4. FURTHER PROPERTIES OF F

Geoghegan discovered the interest in knowing whether or not F is amenable; he conjectured in 1979 (see p. 549 of [GeS]) that F does not contain a non-Abelian free subgroup and that F is not amenable. Brin and Squier proved in [BriS] that F does not contain a non-Abelian free subgroup, but it is still unknown whether or not F is amenable. We first define amenable, and then discuss why the question of amenability of F is so interesting. For further information, see [GriK], [P], or [W].

A discrete group G is *amenable* if there is a left-invariant measure μ on G which is finitely additive and has total measure 1. That is, G is amenable if there is a function $\mu\colon \{\text{subsets of } G\} \to [0,1]$ such that

1) $\mu(gA) = \mu(A)$ for all $g \in G$ and all subsets A of G,
2) $\mu(G) = 1$, and
3) $\mu(A \cup B) = \mu(A) + \mu(B)$ if A and B are disjoint subsets of G.

It is clear from the definition that a finite group is amenable. We will prove by contradiction that the free group $K = \langle a, b\rangle$ is not amenable. Suppose otherwise, and let μ be a finitely additive, left invariant measure on K with finite total measure. Then $\mu(\{1\}) = 0$ since K is infinite. For each $g \in \{a, b, a^{-1}, b^{-1}\}$, let $g* = \{h \in K : h$ has a freely reduced representative beginning with $g\}$. Then $a^{-1}(a*) = (b*) \cup (a*) \cup (b^{-1}*) \cup \{1\}$, so $\mu(a*) = \mu(b*) + \mu(a*) + \mu(b^{-1}*)$ and hence $\mu(b*) = \mu(b^{-1}*) = 0$. Similarly, $\mu(a*) = \mu(a^{-1}*) = 0$. Since

$$K = \{1\} \cup (a^{-1}*) \cup (b*) \cup (a*) \cup (b^{-1}*),$$

$\mu(K) = 0$.

The idea of amenability arose from Banach's paper [Ban], in which he proved that the Monotone Convergence Theorem does not follow from the other axioms of Lebesgue measure. In [N], von Neumann defined amenability (though the term amenable is due to Day [Da]). Von Neumann proved that the free group of rank two is not amenable, and he made the connection between Banach-Tarski paradoxes and nonamenability of the isometry groups. He proved that the class of all amenable groups contains all Abelian groups and all finite groups, and is closed under quotients, subgroups, extensions, and directed unions with respect to inclusion. We call a group an *elementary amenable group* if it is in the smallest class of groups that contains all Abelian and finite groups and is closed under quotients, subgroups, extensions, and directed unions with respect to inclusion.

Following [Da], let EG denote the class of elementary amenable groups, let AG denote the class of amenable discrete groups, and let NF denote the class of groups that do not contain a free subgroup of rank two. Day noted in [Da] that $EG \subset AG$ and $AG \subset NF$ (this follows from [N]), and added that it is not known whether $EG = AG$ or $AG = NF$. The conjecture that $AG = NF$ is known as von Neumann's conjecture or Day's conjecture; it is not stated explicitly in [N] or in [Da].

Olshanskii (see [O]) proved that $AG \neq NF$; Gromov later gave an independent proof in [Gro]. Grigorchuk [Gri1] proved that $EG \neq AG$. However, none of their examples is finitely presented. There are no known finitely presented groups that are in $NF \setminus AG$ or in $AG \setminus EG$. Brin and Squier proved in [BriS, Theorem 3.1] that $F \in NF$ (Corollary 4.9 here). We prove in Theorem 4.10 that F is not an elementary amenable group. If F is amenable, then F is a finitely presented group in $AG \setminus EG$; if F is not amenable, then F is a finitely presented group in $NF \setminus AG$.

One approach to proving that F is not amenable would be to show that $H_b^n(F, \mathbf{R}) \neq 0$ for some positive integer n, where the subscript b indicates bounded cohomology. This was suggested by Grigorchuk in [Gri2], which is a reference for the results in this paragraph. If a group G is amenable, then $H_b^n(G, \mathbf{R}) = 0$ for all positive integers n by Trauber's theorem. Since it is true for any group G that $H_b^0(G, \mathbf{R}) = \mathbf{R}$ and $H_b^1(G, \mathbf{R}) = 0$, the first nontrivial case is $n = 2$. It follows from [DeV] that F' is uniformly perfect. This fact can be used to show that $H_{b,2}^2(F, \mathbf{R}) = 0$. Ghys and Sergiescu have observed that, in fact, $H_b^2(F, \mathbf{R}) = 0$.

THEOREM 4.1. *The commutator subgroup* $[F, F]$ *of* F *consists of all elements in* F *which are trivial in neighborhoods of 0 and 1. Furthermore,* $F/[F, F] \cong \mathbf{Z} \oplus \mathbf{Z}$.

Proof. There exists a group homomorphism $\varphi \colon F \to \mathbf{Z} \oplus \mathbf{Z}$ such that if $f \in F$, then $\varphi(f) = (a, b)$, where the right derivative of f at 0 is 2^a and the left derivative of f at 1 is 2^b. Since $\varphi(A) = (-1, 1)$ and $\varphi(B) = (0, 1)$, φ is surjective. It is easy to see that if K is a group generated by two elements and there exists a surjective group homomorphism from K to $\mathbf{Z} \oplus \mathbf{Z}$, then the kernel of that homomorphism is the commutator subgroup of K. Corollary 2.6 shows that F is generated by A and B, and so $[F, F] = \ker(\varphi)$. This proves Theorem 4.1. $\square$

LEMMA 4.2. *If* $0 = x_0 < x_1 < x_2 < \cdots < x_n = 1$ *and* $0 = y_0 < y_1 < y_2 < \cdots < y_n = 1$ *are partitions of* $[0,1]$ *consisting of dyadic rational numbers, then there exists* $f \in F$ *such that* $f(x_i) = y_i$ *for* $i = 0, \ldots, n$. *Furthermore, if* $x_{i-1} = y_{i-1}$ *and* $x_i = y_i$ *for some* i *with* $1 \leq i \leq n$, *then* f *can be taken to be trivial on the interval* $[x_{i-1}, x_i]$.

Proof. Let m be a positive integer such that $2^m x_i \in \mathbf{Z}$ and $2^m y_i \in \mathbf{Z}$ for $i = 0, \ldots, n$. Let $R = S$ be the $\mathcal{T}$-tree whose leaves consist of the standard dyadic intervals of length 2^{-m}. Let I be the leaf of R whose right endpoint is x_1, and let J be the leaf of S whose right endpoint is y_1. By adjoining carets to R with roots not right of I or adjoining carets to S with roots not right of J, it may be assumed that there are as many leaves in R left of I as there are in S left of J. Continue in this way to enlarge R and S if necessary so that the function f with tree diagram (R, S) maps x_i to y_i for $i = 0, \ldots, n$. This easily proves Lemma 4.2. $\square$

THEOREM 4.3. *Every proper quotient group of* F *is Abelian.*

Proof. Let N be a nontrivial normal subgroup of F. It must be proved that F/N is Abelian.

For this it will be shown in this paragraph that the center of F is trivial. Let f be in the center of F. Since f commutes with B, f and f^{-1} stabilize the fixed point set of B, namely, $[0, \frac{1}{2}] \cup \{1\}$. This implies that $f(\frac{1}{2}) = \frac{1}{2}$. Because every element of F commutes with f, every element of F stabilizes the fixed point set of f. This and Lemma 4.2 easily imply that the fixed point set of f is $[0,1]$. Thus the center of F is trivial.

Because N contains a nontrivial element and the center of F is trivial, N contains a nontrivial commutator of F. Let

$$f = X_0^{b_0} X_1^{b_1} X_2^{b_2} \cdots X_n^{b_n} X_n^{-a_n} \cdots X_2^{-a_2} X_1^{-a_1} X_0^{-a_0}$$

be such an element expressed in normal form. It is easy to see using the map φ in the proof of Theorem 4.1 that $a_0 = b_0$. Let k be the smallest index such that $a_k \neq b_k$. By replacing f by f^{-1} if necessary, it may be assumed that $b_k > a_k$. By replacing f by

$$X_k^{-a_k} \cdots X_2^{-a_2} X_1^{-a_1} X_0^{-a_0} f X_0^{a_0} X_1^{a_1} X_2^{a_2} \cdots X_k^{a_k},$$

it may be assumed that $b_0 = \cdots = b_{k-1} = 0$, $a_0 = \cdots = a_k = 0$, and $b_k > 0$. By replacing f by $X_0^{k-1} f X_0^{1-k}$ it may be assumed that $a_0 = a_1 = b_0 = 0$ and $b_1 > 0$. In this case $(X_0^{-1} f X_0)(X_1^{-1} f X_1)^{-1} = X_2^{b_1} X_1^{-b_1}$. Hence N contains $X_1^{-b}(X_2^b X_1^{-b}) X_1^b = X_1^{-b} X_2^b$ for some positive integer b. Hence N contains

$$X_0X_2^{b-1}\left(X_2(X_1^{-b}X_2^b)X_2^{-1}(X_2^{-b}X_1^b)\right)X_2^{1-b}X_0^{-1} = X_0X_2^{b-1}(X_2X_{b+2}^{-1})X_2^{1-b}X_0^{-1}$$
$$= X_0X_2X_3^{-1}X_0^{-1} = X_1X_2^{-1} = BA^{-1}B^{-1}A\,.$$

Thus F/N is Abelian.

This proves Theorem 4.3. □

LEMMA 4.4. *Let a, b be dyadic rational numbers with $0 \le a < b \le 1$ such that $b-a$ is a power of 2. Then the subgroup of F consisting of all functions with support in $[a,b]$ is isomorphic with F by means of the straightforward linear conjugation.*

Proof. Let $\varphi\colon [a,b] \to [0,1]$ be the linear homeomorphism defined by $\varphi(x) = \frac{1}{b-a}x - \frac{a}{b-a}$. Then $\varphi^{-1}\colon [0,1] \to [a,b]$ is given by $\varphi^{-1}(x) = (b-a)x + a$. The isomorphism from F to the subgroup in question is defined so that for every $f \in F$, $f \mapsto \varphi^{-1}f\varphi$. Where it exists, the derivative of $\varphi^{-1}f\varphi$ is $f'\varphi$. The functions φ and φ^{-1} both map dyadic rational numbers to dyadic rational numbers. Thus f is a function from [0,1] to [0,1] whose points of nondifferentiability are dyadic rational numbers if and only if $\varphi^{-1}f\varphi$ is a function from $[a,b]$ to $[a,b]$ whose points of nondifferentiability are dyadic rational numbers. Lemma 4.4 easily follows. □

THEOREM 4.5. *The commutator subgroup $[F,F]$ of F is a simple group.*

Proof. Let N be a normal subgroup of $[F,F]$ containing a nontrivial element f. According to Theorem 4.1, f is trivial in a neighborhood of 0 and a neighborhood of 1. Theorem 4.1 and Lemma 4.2 easily imply that there exists $g \in [F,F]$ which maps neighborhoods of the intervals $[0,\frac{1}{4}]$ and $[\frac{3}{4},1]$ into these neighborhoods of 0 and 1. Thus gfg^{-1} is a nontrivial function in N whose support lies in $\left[\frac{1}{4},\frac{3}{4}\right]$. According to Lemma 4.4 the subgroup of all functions in F with support in $\left[\frac{1}{4},\frac{3}{4}\right]$ is isomorphic with F. Now Theorem 4.3 shows that N contains the commutator subgroup of the subgroup of F of all functions with support in $\left[\frac{1}{4},\frac{3}{4}\right]$. Thus N contains all functions in F which are trivial in neighborhoods of the intervals $\left[0,\frac{1}{4}\right]$ and $\left[\frac{3}{4},1\right]$. Just as the above function f is conjugated by g into this set of functions, every element of $[F,F]$ is $[F,F]$-conjugate to a function in this set.

This proves Theorem 4.5. □

THEOREM 4.6. *The submonoid of F generated by A, B, B^{-1} is the free product of the submonoid generated by A and the subgroup generated by B.*

Proof. The proof will deal with reduced words in A, B, B^{-1}. Given such a reduced word w, let $\overline{w}$ denote the corresponding element in F. What must be shown is that if w_1 and w_2 are two reduced words in A, B, B^{-1} with $\overline{w_1} = \overline{w_2}$, then $w_1 = w_2$.

Suppose that there exist reduced words w_1, w_2 in A, B, B^{-1} with $\overline{w_1} = \overline{w_2}$ and $w_1 \neq w_2$. Choose such words w_1 and w_2 so that the sum of their lengths is minimal. Suppose that one of w_1 and w_2 ends (on the right) with B and the other ends with B^{-1}. Then $\overline{w_1 B} = \overline{w_2 B}$, $w_1 B \neq w_2 B$, and the sum of the lengths of $w_1 B$ and $w_2 B$ is minimal. Thus by multiplying w_1 and w_2 on the right by an appropriate power of B, it may further be assumed that w_1 ends with A. Because the sum of the lengths of w_1 and w_2 is minimal, w_2 ends with either B or B^{-1}.

There exists a group homomorphism $\varphi\colon F \to \mathbf{Z}$ such that $\varphi(A) = 1$ and $\varphi(B) = 0$. Hence $\varphi(\overline{w_1}) = \varphi(\overline{w_2})$ implies that the number of A's which occur in w_1 equals the number of A's which occur in w_2. Let n be this number of A's. Clearly $n > 0$.

Now note that $A\left(\frac{3}{4}\right) = \frac{1}{2}$ and moreover $A^n\left(\frac{3}{4}\right) = 2^{-n}$. Because B and B^{-1} act trivially on the closed interval $\left[0, \frac{1}{2}\right]$, it follows that $\overline{w_1}\left(\frac{3}{4}\right) = 2^{-n}$.

Suppose that w_2 ends with B. Then w_2 ends with AB^m for some positive integer m. Note that $\frac{1}{2} < B^m\left(\frac{3}{4}\right) < \frac{3}{4}$, and so $\frac{1}{4} < AB^m\left(\frac{3}{4}\right) < \frac{1}{2}$. Again because B and B^{-1} act trivially on the closed interval $\left[0, \frac{1}{2}\right]$, it follows that $\overline{w_2}\left(\frac{3}{4}\right)$ is not a power of 2, contrary to the fact that $\overline{w_1}\left(\frac{3}{4}\right) = 2^{-n}$.

Thus w_2 ends with B^{-1}. Now note that $\frac{7}{8} \leq B^{-1}(x) = A^{-1}(x)$ for every x in the interval $\left[\frac{3}{4}, 1\right]$. Because $\overline{w_2}\left(\frac{3}{4}\right) = 2^{-n}$, it follows that $w_2 = w_3 w_4$, where w_3 and w_4 are reduced words in A, B, B^{-1} with $w_4\left(\frac{3}{4}\right) = \frac{3}{4}$ and w_3 ends with either A or B. If w_3 ends with A, then the argument of the penultimate paragraph shows that $\overline{w_2}\left(\frac{3}{4}\right) = \overline{w_3}\left(\frac{3}{4}\right) = 2^{-n'}$, where n' is the number of A's in w_3. But A occurs in w_4 because $w_4\left(\frac{3}{4}\right) = \frac{3}{4}$, and so $n' < n$. This is impossible, and so w_3 ends with B. The argument of the previous paragraph shows in this case that $\overline{w_2}\left(\frac{3}{4}\right)$ is not even a power of 2. This contradiction completes the proof of the theorem. □

COROLLARY 4.7. *Thompson's group F has exponential growth.*

Theorem 4.8 and Corollary 4.9 were proved in [BriS] for the supergroup of F of orientation-preserving, piecewise-linear homeomorphisms of $\mathbf{R}$ that have slope 1 near $-\infty$ and ∞.

THEOREM 4.8. *Every non-Abelian subgroup of F contains a free Abelian subgroup of infinite rank.*

Proof. Let K be a subgroup of F generated by elements f, g such that $[f, g] \neq 1$. Let $I_1, \dots, I_n$ be the closed intervals in $[0, 1]$ with nonempty interiors such that for every integer k with $1 \leq k \leq n$, if x is an endpoint of I_k, then $f(x) = g(x) = x$ and if x is an interior point of I_k, then either $f(x) \neq x$ or $g(x) \neq x$.

In this paragraph it will be shown for every integer k with $1 \leq k \leq n$ that the endpoints of I_k are cluster points of the K-orbit of every interior point of I_k. Let x be an interior point of I_k. Let y be the greatest lower bound of the K-orbit of x. If y is not the left endpoint of I_k, then either $f(y) \neq y$ or $g(y) \neq y$. Suppose that $f(y) \neq y$. Then either $f(y) < y$ or $f^{-1}(y) < y$. Hence there exists a neighborhood of y such that every element of its image under either f or f^{-1} is less than y. Thus y is the left endpoint of I_k. The same argument applies to least upper bounds. This proves for every integer k with $1 \leq k \leq n$ that the endpoints of I_k are cluster points of the K-orbit of every interior point of I_k.

Let $h_1 = [f, g]$. Just as commutators in F are trivial in neighborhoods of 0 and 1, h_1 is trivial in neighborhoods of the endpoints of I_1. The result of the previous paragraph implies that h_1 is conjugate in K to a function h_2 whose support in I_1 is disjoint from the support of h_1 in I_1. It easily follows that there exists an infinite sequence of functions $h_1, h_2, h_3, \dots$ in K whose supports in I_1 are mutually disjoint. Thus $[h_i, h_j]$ is trivial on I_1 for all positive integers i, j. If $[h_i, h_j] = 1$ for all positive integers i and j, then it is easy to see that $h_1, h_2, h_3, \dots$ form a basis of a free Abelian subgroup of K, as desired.

If $[h_i, h_j] \neq 1$ for some positive integers i and j, then repeat the argument of the previous paragraph with h_1 replaced by this nontrivial commutator $[h_i, h_j]$ and I_1 replaced by some interval I_k on which $[h_i, h_j]$ is not trivial. This process eventually leads to an infinite sequence of functions $h_1, h_2, h_3, \dots$ in K which form a basis of a free Abelian subgroup.

This proves Theorem 4.8. $\square$

COROLLARY 4.9. *Thompson's group F does not contain a non-Abelian free group.*

The next result relies on the paper [C] by Ching Chou.

THEOREM 4.10. *Thompson's group F is not an elementary amenable group.*

Proof. According to (a) of Chou's Proposition 2.2, it suffices to prove that $F \notin EG_\alpha$ for every ordinal α. Since EG_0 consists of finite groups and Abelian groups, it is clear that $F \notin EG_0$, so assume that $\alpha > 0$ and that $F \notin EG_\beta$ for every ordinal $\beta < \alpha$.

If α is a limit ordinal, then there is nothing to prove. Suppose that α is not a limit ordinal. It must be shown that F cannot be constructed from groups in $EG_{\alpha-1}$ as a group extension or as a direct union.

First consider group extensions. Suppose that F contains a normal subgroup N such that N, $F/N \in EG_{\alpha-1}$. Since $F \notin EG_{\alpha-1}$, N is nontrivial. Theorem 4.3 implies that $[F, F] \subset N$. Now Theorem 4.1 and Lemma 4.4 easily imply that N contains a subgroup isomorphic with F. Proposition 2.1 of [C] states that subgroups of groups in $EG_{\alpha-1}$ are also in $EG_{\alpha-1}$. Thus $F \in EG_{\alpha-1}$, contrary to hypothesis. This proves that F cannot be constructed from $EG_{\alpha-1}$ as a group extension.

Second consider direct unions. Suppose that F is a direct union of groups in $EG_{\alpha-1}$. This is clearly impossible because F is finitely generated.

This proves Theorem 4.10. $\square$

We next show that F is a totally ordered group (this also follows from [BriS]). Define the set of *order positive* elements of F to be the set P of functions $f \in F$ such that there exists a subinterval $[a, b]$ of $[0, 1]$ on which the derivative of f is less than 1 and $f(x) = x$ for $0 \leq x \leq a$. It is easy to see that the positive elements of F are indeed order positive. It is clear that $F = P^{-1} \cup \{1\} \cup P$. It is easy to see that P is closed under multiplication and $f^{-1}Pf \subset P$ for every $f \in F$. This proves Theorem 4.11.

THEOREM 4.11. *Thompson's group F is a totally ordered group.*

§5. THOMPSON'S GROUP T

The material in this section is mainly from unpublished notes of Thompson [T1].

Consider S^1 as the interval $[0, 1]$ with the endpoints identified. Then T is the group of piecewise linear homeomorphisms from S^1 to itself that map images of dyadic rational numbers to images of dyadic rational numbers