

Zeitschrift: L'Enseignement Mathématique
Herausgeber: Commission Internationale de l'Enseignement Mathématique
Band: 42 (1996)
Heft: 3-4: L'ENSEIGNEMENT MATHÉMATIQUE

Artikel: INTRODUCTORY NOTES ON RICHARD THOMPSON'S GROUPS
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Kapitel: §2. Tree diagrams
DOI: <https://doi.org/10.5169/seals-87877>

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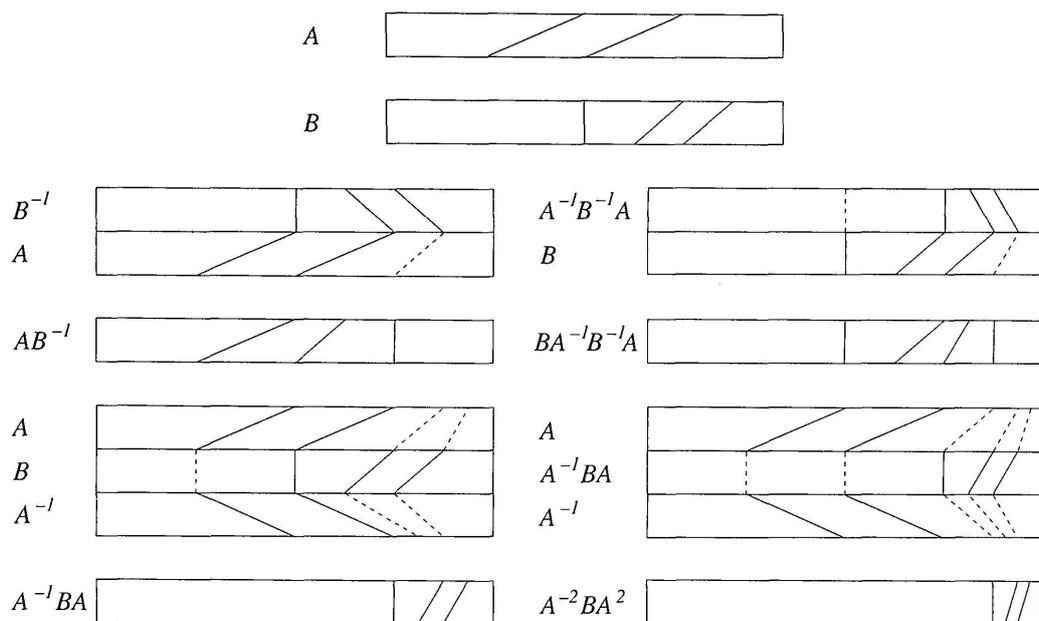


FIGURE 1
The rectangle diagrams of some elements of F

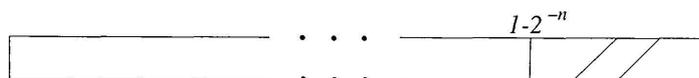


FIGURE 2
The rectangle diagram of X_n

§2. TREE DIAGRAMS

The notion of tree diagram is developed in this section. Tree diagrams are useful for describing functions in F ; we first encountered them in [Bro1].

Define an *ordered rooted binary tree* to be a tree S such that i) S has a root v_0 , ii) if S consists of more than v_0 , then v_0 has valence 2, and iii) if v is a vertex in S with valence greater than 1, then there are exactly two edges $e_{v,L}$, $e_{v,R}$ which contain v and are not contained in the geodesic from v_0 to v . The edge $e_{v,L}$ is called a *left edge* of S , and $e_{v,R}$ is called a *right edge* of S . Vertices with valence 0 (in case of the trivial tree) or 1 in S will be called *leaves* of S . There is a canonical left-to-right linear ordering on the leaves of S . The *right side* of S is the maximal arc of right edges in S which begins at the root of S . The *left side* of S is defined analogously.

An *isomorphism* of ordered rooted binary trees is an isomorphism of rooted trees which takes left edges to left edges and right edges to right edges. An *ordered rooted binary subtree* S' of an ordered rooted binary tree S is an

ordered rooted binary tree which is a subtree of S whose left edges are left edges of S , whose right edges are right edges of S , but whose root need not be the root of S .

EXAMPLE 2.1. The right side of the ordered rooted binary tree in Figure 3 is highlighted. Its leaves are labeled $0, \dots, 5$ in order.

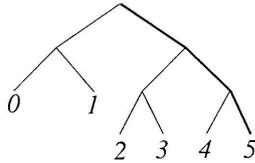


FIGURE 3

An ordered rooted binary tree with 6 leaves

Define a *standard dyadic interval* in $[0, 1]$ to be an interval of the form $[\frac{a}{2^n}, \frac{a+1}{2^n}]$, where a, n are nonnegative integers with $a \leq 2^n - 1$.

There is a *tree of standard dyadic intervals*, \mathcal{T} , which is defined as follows. The vertices of \mathcal{T} are the standard dyadic intervals in $[0, 1]$. An edge of \mathcal{T} is a pair (I, J) of standard dyadic intervals I and J such that either I is the left half of J , in which case (I, J) is a left edge, or I is the right half of J , in which case (I, J) is a right edge. It is easy to see that \mathcal{T} is an ordered rooted binary tree. The tree of standard dyadic intervals is shown in Figure 4.

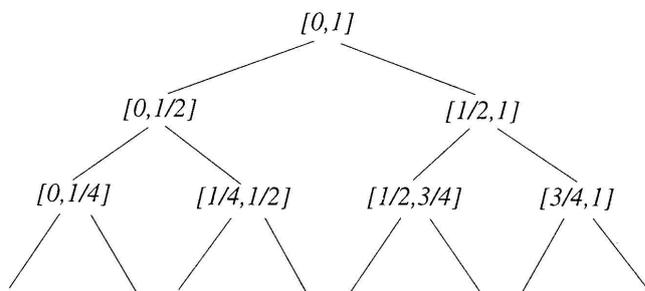


FIGURE 4

The tree \mathcal{T} of standard dyadic intervals

Define a \mathcal{T} -tree to be a finite ordered rooted binary subtree of \mathcal{T} with root $[0,1]$. Call the \mathcal{T} -tree with just one vertex the *trivial \mathcal{T} -tree*. For every nonnegative integer n , let \mathcal{T}_n be the \mathcal{T} -tree with $n + 1$ leaves whose right side has length n . \mathcal{T}_3 is shown in Figure 5.

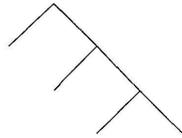


FIGURE 5
The \mathcal{T} -tree \mathcal{T}_3

Define a *caret* to be an ordered rooted binary subtree of \mathcal{T} with exactly two edges. Every caret has the form of the rooted tree in Figure 6.



FIGURE 6
A caret

A partition $0 = x_0 < x_1 < x_2 < \cdots < x_n = 1$ of $[0, 1]$ determines intervals $[x_{i-1}, x_i]$ for $i = 1, \dots, n$ which are called the *intervals of the partition*. A partition of $[0, 1]$ is called a *standard dyadic partition* if and only if the intervals of the partition are standard dyadic intervals.

It is easy to see that the leaves of a \mathcal{T} -tree are the intervals of a standard dyadic partition. Conversely, the intervals of a standard dyadic partition determine finitely many vertices of \mathcal{T} , and it is easy to see that these vertices are the leaves of their convex hull, which is a \mathcal{T} -tree. Thus there is a canonical bijection between standard dyadic partitions and \mathcal{T} -trees.

LEMMA 2.2. *Let $f \in F$. Then there exists a standard dyadic partition $0 = x_0 < x_1 < x_2 < \cdots < x_n = 1$ such that f is linear on every interval of the partition and $0 = f(x_0) < f(x_1) < f(x_2) < \cdots < f(x_n) = 1$ is a standard dyadic partition.*

Proof. Choose a partition P of $[0, 1]$ whose partition points are dyadic rational numbers such that f is linear on every interval of P . Let $[a, b]$ be an interval of P . Suppose that the derivative of f on $[a, b]$ is 2^{-k} . Let m be an integer such that $m \geq 0$, $m + k \geq 0$, $2^m a \in \mathbf{Z}$, $2^m b \in \mathbf{Z}$, $2^{m+k} f(a) \in \mathbf{Z}$, and $2^{m+k} f(b) \in \mathbf{Z}$. Then $a < a + \frac{1}{2^m} < a + \frac{2}{2^m} < a + \frac{3}{2^m} < \cdots < b$ partitions $[a, b]$ into standard dyadic intervals, and $f(a) < f(a) + \frac{1}{2^{m+k}} < f(a) + \frac{2}{2^{m+k}} < f(a) + \frac{3}{2^{m+k}} < \cdots < f(b)$ partitions $[f(a), f(b)]$ into standard dyadic intervals. This easily proves Lemma 2.2. \square

Formally, a *tree diagram* is an ordered pair (R, S) of \mathcal{T} -trees such that R and S have the same number of leaves. This is rendered diagrammatically as follows :

$$R \rightarrow S.$$

The tree R is called the *domain tree* of the diagram, and S is called the *range tree* of the diagram.

Suppose given $f \in F$. Lemma 2.2 shows that there exist standard dyadic partitions P and Q such that f is linear on the intervals of P and maps them to the intervals of Q . To f is associated the tree diagram (R, S) , where R is the \mathcal{T} -tree corresponding to P and S is the \mathcal{T} -tree corresponding to Q .

Because P and Q are not unique, there are many tree diagrams associated to f . Given one tree diagram (R, S) for f , another can be constructed by adjoining carets to R and S as follows. Let I be the n^{th} leaf of R for some positive integer n , and let J be the n^{th} leaf of S . Let I_1, I_2 be the leaves in order of the caret C with root I , and let J_1, J_2 be the leaves in order of the caret D with root J . Because f is linear on I and $f(I) = J$, it follows that $f(I_1) = J_1$ and $f(I_2) = J_2$. Thus (R', S') is a tree diagram for f , where $R' = R \cup C$ and $S' = S \cup D$.

In the other direction, if there exists a positive integer n such that the n^{th} and $(n+1)^{\text{th}}$ leaves of R , respectively S , are the vertices of a caret C , respectively D , then deleting all of C and D but the roots from R and S leads to a new tree diagram for f . If there do not exist such carets C, D in R, S , then the tree diagram (R, S) is said to be *reduced*.

In this paragraph it will be shown that there is exactly one reduced tree diagram for f . Suppose that (R, S) is a reduced tree diagram for f . It is easy to see that if I is a standard dyadic interval which is either a leaf of R or not in R , then $f(I)$ is a standard dyadic interval and f is linear on I . Conversely, if I is a standard dyadic interval such that $f(I)$ is a standard dyadic interval and f is linear on I , then I is either a leaf of R or not in R because (R, S) is reduced. Thus R is the unique \mathcal{T} -tree such that a standard dyadic interval I is either a leaf of R or not in R if and only if $f(I)$ is a standard dyadic interval and f is linear on I . This gives uniqueness of reduced tree diagrams.

Furthermore, if (R, S) is a tree diagram, then it is clear that there exists $f \in F$ such that f is linear on every leaf of R and f maps the leaves of R to the leaves of S .

Thus there is a canonical bijection between F and the set of reduced tree diagrams.

EXAMPLE 2.3. Figure 7 shows the reduced tree diagrams for A and B .

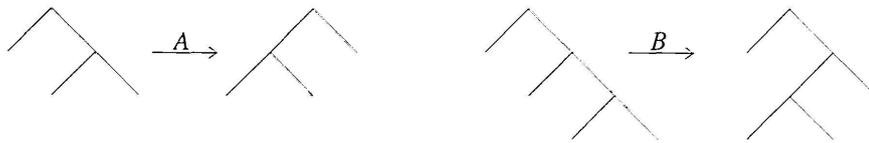


FIGURE 7

The reduced tree diagrams for A and B

From Figure 2 it is not difficult to see that, for $n \geq 0$, the reduced tree diagram for X_n is the tree diagram in Figure 8.

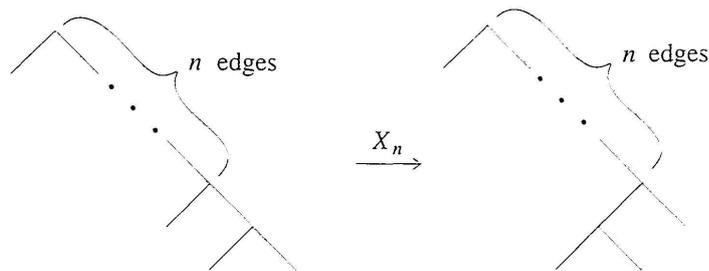


FIGURE 8

The reduced tree diagram for X_n

It is easy to see that if (Q, R) is a tree diagram for a function f in F and (R, S) is a tree diagram for a function g in F , then (Q, S) is a tree diagram for gf .

The following definition prepares for Theorem 2.5, which makes the correspondence between functions in F and tree diagrams more precise. Define the *exponents* of a \mathcal{T} -tree S as follows. Let I_0, \dots, I_n be the leaves of S in order. For every integer k with $0 \leq k \leq n$ let a_k be the length of the maximal arc of left edges in S which begins at I_k and which does not reach the right side of S . Then a_k is the k^{th} exponent of S .

EXAMPLE 2.4. Let S be the \mathcal{T} -tree shown in Figure 9.

The leaves of S are labeled $0, \dots, 9$ in order, and the exponents of S in order are $2, 1, 0, 0, 1, 2, 0, 0, 0, 0$.

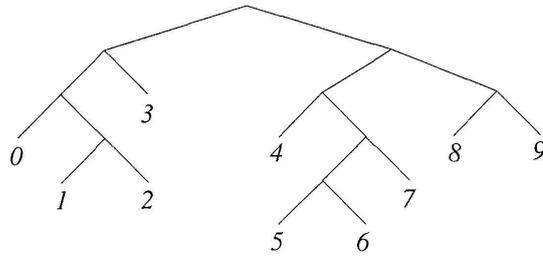


FIGURE 9
The \mathcal{T} -tree S

THEOREM 2.5. *Let R, S be \mathcal{T} -trees with $n+1$ leaves for some nonnegative integer n . Let a_0, \dots, a_n be the exponents of R , and let b_0, \dots, b_n be the exponents of S . Then the function in F with tree diagram (R, S) is $X_0^{b_0} X_1^{b_1} X_2^{b_2} \dots X_n^{b_n} X_n^{-a_n} \dots X_2^{-a_2} X_1^{-a_1} X_0^{-a_0}$. The tree diagram (R, S) is reduced if and only if i) if the last two leaves of R lie in a caret, then the last two leaves of S do not lie in a caret and ii) for every integer k with $0 \leq k < n$, if $a_k > 0$ and $b_k > 0$ then either $a_{k+1} > 0$ or $b_{k+1} > 0$.*

Proof. To prove the first statement of the theorem, by composing functions it suffices to prove that the function with tree diagram (R, \mathcal{T}_n) is $X_n^{-a_n} \dots X_2^{-a_2} X_1^{-a_1} X_0^{-a_0}$.

The proof of this will proceed by induction on $a = \sum_{i=0}^n a_i$. If $a = 0$, then $R = \mathcal{T}_n$, and the result is clear. Now suppose that $a > 0$ and that the result is true for smaller values of a . Let m be the smallest index such that $a_m > 0$. Then there are ordered rooted binary subtrees R_1, R_2, R_3 of R such that R has the form of the tree at the left of Figure 10.

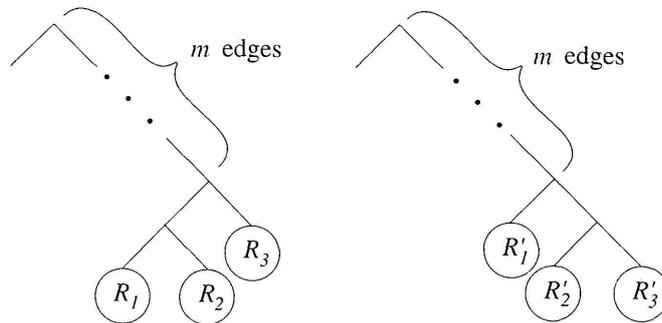


FIGURE 10
The \mathcal{T} -trees R and R'

Let R' be the \mathcal{T} -tree shown at the right of Figure 10, where R'_1, R'_2, R'_3 are isomorphic with R_1, R_2, R_3 as ordered rooted binary trees. According to Example 2.3, the function with tree diagram (R, R') is X_m^{-1} . If a'_0, \dots, a'_n are the exponents of R' , then $a'_m = a_m - 1$ and $a'_k = a_k$ if $k \neq m$. Thus

the induction hypothesis applies to R' , and so the function with tree diagram (R', \mathcal{T}_n) is $X_n^{-a'_n} \cdots X_2^{-a'_2} X_1^{-a'_1} X_0^{-a'_0}$. Again by composing functions, it follows that the function with tree diagram (R, \mathcal{T}_n) is $X_n^{-a_n} \cdots X_2^{-a_2} X_1^{-a_1} X_0^{-a_0}$, as desired.

The second statement of the theorem is now easy to prove.

This proves Theorem 2.5. \square

COROLLARY 2.6. *Thompson's group F is generated by A and B .*

COROLLARY-DEFINITION 2.7. *Every nontrivial element of F can be expressed in unique normal form*

$$X_0^{b_0} X_1^{b_1} X_2^{b_2} \cdots X_n^{b_n} X_n^{-a_n} \cdots X_2^{-a_2} X_1^{-a_1} X_0^{-a_0},$$

where $n, a_0, \dots, a_n, b_0, \dots, b_n$ are nonnegative integers such that i) exactly one of a_n and b_n is nonzero and ii) if $a_k > 0$ and $b_k > 0$ for some integer k with $0 \leq k < n$, then $a_{k+1} > 0$ or $b_{k+1} > 0$. Furthermore, every such normal form function in F is nontrivial.

The functions in F of the form $X_0^{b_0} X_1^{b_1} X_2^{b_2} \cdots X_n^{b_n}$ with $b_k \geq 0$ for $k = 0, \dots, n$ will be called *positive*. The positive elements of F are exactly those with tree diagrams having domain tree \mathcal{T}_n for some nonnegative integer n . Inverses of positive elements will be called *negative*.

LEMMA 2.8. *The set of positive elements of F is closed under multiplication.*

Proof. Let f and g be positive elements of F . Let (\mathcal{T}_m, R) , respectively (\mathcal{T}_n, S) , be tree diagrams for f , respectively g . If the right side of S has length k , then it is easy to see that fg has a tree diagram with domain tree $\mathcal{T}_{n+\max\{m-k, 0\}}$. Thus fg is positive. This proves Lemma 2.8. \square

Fordham [Fo] gives a linear-time algorithm that takes as input the reduced tree diagram representing an element of Thompson's group F and gives as output the minimal length of a word in generators A and B representing that element. The algorithm can be modified to actually construct one, or all, minimal representatives. Fordham assigns a type to each caret of the tree pair; the minimal length is a simple function of the type sequences of the two trees.