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$\pi_A : Y \rightarrow A$  we set  $A_\Theta(s) := \pi_A \circ \mu^{-1}(\mathcal{H}_{1\Theta}(s)) \subset A$ . The set  $A_\Theta(s)$  is contained in an “affine” subspace of  $A$  of the form  $a_1 a_*(s) A^{q-k}$  where  $a_1 a_*(s) \in A$  and  $A^{q-k}$  is a  $q-k$ -dimensional subgroup of  $A$  (see Sections 3 and 4 of [L2]). We denote the restriction of  $dv_A$  to  $A^{q-k}$  by  $dv_{A^{q-k}}$ ; for  $k = q$  we have  $A^0 = e$  and we set  $dv_{A^0} \equiv 1$ . By Lemma 3.3 we have (for  $k$  equal to the number of elements of  $\Theta$ )

$$\text{Vol}(V_E^{n-k}(s)) \prec \int_{\mu^{-1}(\mathcal{H}_{1\Theta}(s))} \rho(a)^{-1} dv_U \wedge dv_Z \wedge dv_{A^{q-k}}.$$

Since the horospherical piece  $\mathcal{H}_{1\Theta}(s)$  is part of a Siegel set  $\mathcal{S}_{\omega,\tau}$  with  $\omega$  relatively compact (and hence of finite volume) in  $UM$  we get

$$\begin{aligned} \int_{\mu^{-1}(\mathcal{H}_{1\Theta}(s))} \rho(a)^{-1} dv_U \wedge dv_Z \wedge dv_{A^{q-k}} &\prec \\ &\prec \int_{\omega} dv_U \wedge dv_Z \int_{A_\Theta(s)} \rho(a)^{-1} dv_{A^{q-k}} \prec \int_{A_\Theta(s)} \rho(a)^{-1} dv_{A^{q-k}}. \end{aligned}$$

Also by definition of a Siegel set we have  $\alpha(a) \geq \tau \succ 1$  for all  $\alpha \in \Delta$ . Moreover, the computations in the proof of Lemma 4.1 (and Lemma 3.5) in [L2] show that for all  $\alpha \in \Theta$  one has  $\alpha(a_1 a_*(s)) \succ e^{\mu_\alpha s}$  with  $\mu_\alpha > 0$ . Hence, as  $\Theta \subset \Delta$  is not empty and since  $\rho = \sum_{\alpha \in \Delta} c_\alpha \alpha$  ( $c_\alpha > 0$ ), there is a uniform constant  $c > 0$  such that  $\rho(a)^{-1} \prec e^{-cs}$  for all  $a \in A_\Theta(s)$ . As noted above the set  $A_\Theta(s)$  is contained in a  $(q-k)$ -dimensional affine cone in  $A$ . It is similar (in the sense of Euclidean geometry) to  $A_\Theta(0)$  with similarity factor  $s$  (see the proof of Lemma 4.1 in [L2]). Hence we eventually get  $\int_{A_\Theta(s)} dv_{A^{q-k}} \prec s^{q-k}$  and the Lemma follows.  $\square$

#### 4. A NEW PROOF OF THE GAUSS-BONNET FORMULA

In this section we present a new simplified proof of the Gauss-Bonnet theorem for higher rank locally symmetric spaces.

**THEOREM 4.1.** *Let  $X$  be a Riemannian symmetric space of noncompact type and  $\mathbb{R}$ -rank  $\geq 2$  and let  $\Gamma$  be an irreducible, torsion-free (non-uniform) lattice in the group of isometries of  $X$ . Then for the locally symmetric space  $V = \Gamma \backslash X$  the Gauss-Bonnet formula holds :*

$$\chi(V) = \int_V \Psi dv.$$

*Proof.* By Proposition 2.2 there is an exhaustion  $V = \bigcup_{s \geq 0} V(s)$  of  $V$  by Riemannian polyhedra  $V(s)$ . Each polyhedron  $V(s)$  in this exhaustion is equipped with the Riemannian metric induced by the one of  $V$ . Proposition 1.1 applied to  $V(s)$  yields

$$\left| (-1)^n \chi'(V(s)) - \int_{V(s)} \Psi dv \right| \prec \sum_{k=1}^q \sum_E \int_{V_E^{n-k}(s)} \int_{O(p)} \|\Psi_{E,k}\| d\omega_{k-1} dv_E(p)$$

where  $q = \dim A$  is the  $\mathbb{Q}$ -rank of  $\mathbf{G}$  (see Section 2.1) and where the index  $E$  runs through a finite set. As we remarked in Section 1 the function  $\Psi_{E,k}$  is locally computable from the components of the metric and the curvature tensor of  $V(s)$  and from the components of the second fundamental form of  $V_E^{n-k}(s)$  in  $V(s)$ . The fact that  $V$  is locally symmetric together with Lemma 3.2 thus implies that  $\|\Psi_{E,k}\| \prec 1$  for all  $E, k$ . Using Lemma 3.4 we conclude that

$$\left| (-1)^n \chi'(V(s)) - \int_{V(s)} \Psi dv \right| \prec \sum_{k,E} \text{Vol}(V_E^{n-k}(s)) \prec e^{-cs} \sum_{k=1}^q s^{q-k}.$$

By Proposition 2.3 we have  $\chi'(V(s)) = \chi(V)$ . The polyhedra  $V(s)$  exhaust  $V$  and  $\chi(V)$  is an integer; hence  $(-1)^n \chi(V) = \int_{V(s)} \Psi dv$  for sufficiently large  $s$ . Finally, for  $n$  odd  $\Psi \equiv 0$  by definition (see [AW]) and the claimed formula follows.  $\square$

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