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$\pi_A : Y \to A$ we set $A_\Theta(s) := \pi_A \circ \mu^{-1}\big(\mathcal{H}_{1\Theta}(s)\big) \subset A$. The set $A_\Theta(s)$ is contained in an "affine" subspace of A of the form $a_1 a_*(s) A^{q-k}$ where $a_1 a_*(s) \in A$ and A^{q-k} is a $q-k$-dimensional subgroup of A (see Sections 3 and 4 of [L2]). We denote the restriction of dv_A to A^{q-k} by $dv_{A^{q-k}}$; for $k = q$ we have $A^0 = e$ and we set $dv_{A^0} \equiv 1$. By Lemma 3.3 we have (for k equal to the number of elements of Θ)

$$\mathrm{Vol}\big(V_E^{n-k}(s)\big) \prec \int_{\mu^{-1}(\mathcal{H}_{1\Theta}(s))} \rho(a)^{-1}\, dv_U \wedge dv_Z \wedge dv_{A^{q-k}}\,.$$

Since the horospherical piece $\mathcal{H}_{1\Theta}(s)$ is part of a Siegel set $\mathcal{S}_{\omega,\tau}$ with ω relatively compact (and hence of finite volume) in UM we get

$$\int_{\mu^{-1}(\mathcal{H}_{1\Theta}(s))} \rho(a)^{-1} dv_U \wedge dv_Z \wedge dv_{A^{q-k}} \prec$$

$$\prec \int_\omega dv_U \wedge dv_Z \int_{A_\Theta(s)} \rho(a)^{-1} dv_{A^{q-k}} \prec \int_{A_\Theta(s)} \rho(a)^{-1} dv_{A^{q-k}}\,.$$

Also by definition of a Siegel set we have $\alpha(a) \geq \tau \succ 1$ for all $\alpha \in \Delta$. Moreover, the computations in the proof of Lemma 4.1 (and Lemma 3.5) in [L2] show that for all $\alpha \in \Theta$ one has $\alpha\big(a_1 a_*(s)\big) \succ e^{\mu_\alpha s}$ with $\mu_\alpha > 0$. Hence, as $\Theta \subset \Delta$ is not empty and since $\rho = \sum_{\alpha\in\Delta} c_\alpha \alpha\, (c_\alpha > 0)$, there is a uniform constant $c > 0$ such that $\rho(a)^{-1} \prec e^{-cs}$ for all $a \in A_\Theta(s)$. As noted above the set $A_\Theta(s)$ is contained in a $(q-k)$-dimensional affine cone in A. It is similar (in the sense of Euclidean geometry) to $A_\Theta(0)$ with similarity factor s (see the proof of Lemma 4.1 in [L2]). Hence we eventually get $\int_{A_\Theta(s)} dv_{A^{q-k}} \prec s^{q-k}$ and the Lemma follows. $\square$

4. A NEW PROOF OF THE GAUSS-BONNET FORMULA

In this section we present a new simplified proof of the Gauss-Bonnet theorem for higher rank locally symmetric spaces.

THEOREM 4.1. *Let X be a Riemannian symmetric space of noncompact type and $\mathbb{R}$-rank ≥ 2 and let Γ be an irreducible, torsion-free (non-uniform) lattice in the group of isometries of X. Then for the locally symmetric space $V = \Gamma\backslash X$ the Gauss-Bonnet formula holds:*

$$\chi(V) = \int_V \Psi\, dv\,.$$

Proof. By Proposition 2.2 there is an exhaustion $V = \bigcup_{s \geq 0} V(s)$ of V by Riemannian polyhedra $V(s)$. Each polyhedron $V(s)$ in this exhaustion is equipped with the Riemannian metric induced by the one of V. Proposition 1.1 applied to $V(s)$ yields

$$\left|(-1)^n \chi'\big(V(s)\big) - \int_{V(s)} \Psi\, dv\right| \prec \sum_{k=1}^{q} \sum_{E} \int_{V_E^{n-k}(s)} \int_{O(p)} \|\Psi_{E,k}\|\, d\omega_{k-1}\, dv_E(p)$$

where $q = \dim A$ is the $\mathbb{Q}$-rank of $\mathbf{G}$ (see Section 2.1) and where the index E runs through a finite set. As we remarked in Section 1 the function $\Psi_{E,k}$ is locally computable from the components of the metric and the curvature tensor of $V(s)$ and from the components of the second fundamental form of $V_E^{n-k}(s)$ in $V(s)$. The fact that V is locally symmetric together with Lemma 3.2 thus implies that $\|\Psi_{E,k}\| \prec 1$ for all E, k. Using Lemma 3.4 we conclude that

$$\left|(-1)^n \chi'\big(V(s)\big) - \int_{V(s)} \Psi dv\right| \prec \sum_{k,E} \mathrm{Vol}\big(V_E^{n-k}(s)\big) \prec e^{-cs} \sum_{k=1}^{q} s^{q-k}.$$

By Proposition 2.3 we have $\chi'\big(V(s)\big) = \chi(V)$. The polyhedra $V(s)$ exhaust V and $\chi(V)$ is an integer; hence $(-1)^n \chi(V) = \int_{V(s)} \Psi\, dv$ for sufficiently large s. Finally, for n odd $\Psi \equiv 0$ by definition (see [AW]) and the claimed formula follows. □

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