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Each polyhedron  $V(s)$  is homotopically equivalent to  $V$ . More precisely we have

**PROPOSITION 2.3.** *For every sufficiently large  $s$  the locally symmetric space  $V$  is homeomorphic to the interior of the polyhedron  $V(s)$  in  $V$ , and  $V(s)$  is a strong deformation retract of  $V$ .*

For the proof see [L3], Theorems 5.2 and 5.5.

### 3. ESTIMATES FOR THE BOUNDARY SUBPOLYHEDRA

We wish to apply Proposition 1.1 to the polyhedra  $V(s)$  in the above exhaustion and then take the limit for  $s \rightarrow \infty$ . To that end we need estimates for the second fundamental forms and the volumes of the (lower dimensional) boundary polyhedra.

For each Siegel set  $\mathcal{S}_i := q_i \mathcal{S}$  which is part of the fundamental set  $\Omega$  we have its truncated part

$$\mathcal{S}_i(s) := \mathcal{S}_i - \bigcup_{\alpha \in \Delta} (\mathcal{B}_{i\alpha}(s) \cap \mathcal{S}_i).$$

The top dimensional boundary faces of  $\mathcal{S}_i(s)$  in  $\mathcal{S}_i$  (resp. of  $\Omega(s)$  in  $\Omega$ ) are subsets of horospheres:

$$\mathcal{H}_{i\alpha}(s) := \{\tau_\alpha^{-1} \tilde{h}_{i\alpha} = -s\} \cap \mathcal{S}_i(s), \quad \alpha \in \Delta.$$

The “horospherical” pieces  $\mathcal{H}_{i\alpha}(s)$  together with their  $\Gamma$ -translates form the boundary of the manifold with corners  $X(s)$  in  $X$ . For any nonempty subset  $\Theta$  of  $\Delta$  we set

$$\mathcal{H}_{i\Theta}(s) := \bigcap_{\alpha \in \Theta} \mathcal{H}_{i\alpha}(s) \subset \mathcal{S}_i(s).$$

The various boundary subpolyhedra of  $V(s)$  are then unions of projections of the pieces  $\mathcal{H}_{i\Theta}(s)$  under the canonical projection  $\pi : X \rightarrow V$ . More precisely, as explained in Section 2, for any subset  $\Theta \subset \Delta$ , we have the equivalence relation on the set  $I = \{1, \dots, m\}$

$$j \sim_\Theta l \quad \text{if and only if} \quad \Gamma q_j P_\Theta = \Gamma q_l P_\Theta$$

(the  $q_i$  are as in Proposition 2.1). This relation  $\sim_\Theta$  induces a partition,  $I(\Theta)$ , of the set  $I$  whose components will be denoted by  $E$ . Let  $n = \dim X = \dim V$ , let  $k$  be the cardinality of  $\Theta$  and let  $E \in I(\Theta)$ . Then  $V_E^{n-k}(s) := \pi(\bigcup_{i \in E} \mathcal{H}_{i\Theta}(s))$  is a  $(n - k)$ -dimensional boundary polyhedron of  $V(s)$ ; and moreover, any boundary polyhedron arises in this way (see [L3] §4).

REMARK. The minimal possible dimension which occurs is  $n - q$  where  $q$  is the  $\mathbb{Q}$ -rank of  $\mathbf{G}$ . It is also interesting to note (though not needed below) that the outer angles are isomorphic to  $\mathbb{Q}$ -Weyl chambers and their walls at infinity.

We shall use the following well-known fact about Jacobi fields in symmetric spaces (see [K] Theorem 2.2.9). A Jacobi field along a geodesic ray is called *stable* if its length is bounded.

LEMMA 3.1. *Let  $r : [0, \infty) \rightarrow X$  be a unit-speed geodesic ray in the symmetric space  $X$  (of noncompact type). Set  $p = r(0)$ . Then the unique stable Jacobi field  $J_u(s)$  along  $r(s)$  with  $J_u(0) = u \in T_p X$  can be written as*

$$J_u(s) = \sum_j e^{-\lambda_j s} a_j E_j(s)$$

where  $\{E_j(s)\}$  is an orthonormal frame of parallel fields along  $r$ , the  $\lambda_j$  are non-negative (uniform) constants and  $u = \sum_j a_j E_j(0)$ .

LEMMA 3.2. *Let  $s \geq 0$ . The second fundamental forms of every boundary polyhedron  $V_E^{n-k}(s)$  with respect to outer angles in  $V(s)$  are uniformly bounded by a constant independent of  $E, k$  and  $s$ .*

*Proof.* Since the claim is local we can work in the universal covering space  $X$ . As we noted above the preimage of  $V_E^{n-k}(s)$  in  $X$  under the projection  $\pi$  is the union of a finite number of horospherical sets

$$\mathcal{H}_{i\Theta}(s) = \bigcap_{\alpha \in \Theta} \mathcal{H}_{i\alpha}(s) \subset \bigcap_{\alpha \in \Theta} \{\tau_\alpha^{-1} \tilde{h}_{i\alpha} = -s\},$$

where  $\Theta$  is a subset of  $\Delta$  with  $k$  elements. The (inner) unit normal field of the horosphere  $\{\tau_\alpha^{-1} \tilde{h}_{i\alpha} = -s\}$  is given by  $Z_{i\alpha} := -\text{grad } \tilde{h}_{i\alpha}$  (see e.g. [HI] Proposition 3.1). Using  $d\pi$  any element in the outer angle  $O(\pi(p))$  of  $V_E^{n-k}(s)$  at a point  $\pi(p) \in V_E^{n-k}(s)$  can then be identified with a positive linear combination (of norm 1) of the radial fields  $Z_{i\alpha}(p)$ ,  $\alpha \in \Theta$ . It therefore suffices to show that for any pair  $(i, \alpha)$  the second fundamental form of  $V_E^{n-k}(s)$  relative to  $d\pi Z_{i\alpha}$  is uniformly bounded. We fix  $i$  and  $\alpha$  and write  $Z$  for  $Z_{i\alpha}$ . For  $p \in X$  let  $\langle \cdot, \cdot \rangle_p$  denote the Riemannian metric of  $X$  at  $p$ . Let  $u, v \in T_p X$  be such that  $d\pi(u), d\pi(v) \in T_{\pi(p)} V_E^{n-k}(s)$ . Using the above identifications the second fundamental form of  $V_E^{n-k}(s) \subset V(s)$  with respect to  $Z$  can be written as

$$\Pi_Z(u, v)(p) = \langle D_u Z, v \rangle_p.$$

According to [HI], Proposition 3.1, we have  $D_u Z(p) = J'_u(0)$  where  $J_u$  is the stable Jacobi field along the (unique) geodesic ray, say  $r$ , in  $X$  which joins  $p$  to  $c_{i\alpha}(\infty) \in \partial_\infty X$  and with initial value  $J_u(0) = u$ . By Lemma 3.1 there are orthonormal parallel fields  $E_j(s)$  along  $r$  and constants  $\lambda_j \geq 0$  such that  $J_u(s) = \sum_j e^{-\lambda_j s} a_j E_j(s)$  with  $u = \sum_j a_j E_j(0)$ . Consequently we get  $J'_u(0) = -\sum_j \lambda_j a_j E_j(0)$  and finally, for  $v = \sum_j b_j E_j(0)$ ,  $|\Pi_Z(u, v)(p)| = |-\sum_j \lambda_j a_j b_j| \prec \|u\| \|v\|$ .  $\square$

We next estimate the volumes of the boundary polyhedra. Recall from Section 2.1 the parametrization of  $X$  by horocyclic coordinates

$$\mu: Y = U \times Z \times A \longmapsto X; (u, \tau(m), a) \longmapsto uma \cdot x_0.$$

Let  $dx^2$  be the  $G$ -invariant Riemannian metric on  $X$  induced by the Cartan-Killing form of the Lie algebra  $\mathfrak{g}$  of  $G$  and let  $dz^2$  be the invariant metric on  $Z$ . Further let  $da^2$  (resp.  $du^2$ ) be the left-invariant metric on  $A$  (resp.  $U$ ). Finally set  $dy^2 := \mu^* dx^2$ .

LEMMA 3.3. *Let  $dv_Y$ ,  $dv_U$ ,  $dv_Z$  and  $dv_A$  denote the volume elements of the metrics  $dy^2$ ,  $du^2$ ,  $dz^2$  and  $da^2$ . Then at the point  $(u, z, a) \in Y$  we have*

$$2^e dv_Y = \rho(a)^{-1} dv_U \wedge dv_Z \wedge dv_A$$

where  $e = \frac{1}{2} \dim U$  and  $\rho$  is the sum of all positive roots (counted with multiplicity); it can be written in the form  $\rho = \sum_{\alpha \in \Delta} c_\alpha \alpha$ ,  $c_\alpha > 0$ .

For the proof see [B2] Corollary 4.4.

LEMMA 3.4. *For the  $(n - k)$ -dimensional volume of each boundary polyhedron  $V_E^{n-k}(s)$  of  $V(s)$  one has the estimate*

$$\text{Vol}(V_E^{n-k}(s)) \prec s^{q-k} e^{-cs},$$

where  $q = \dim A$  is the  $\mathbb{Q}$ -rank of  $\mathbf{G}$  and  $c > 0$  is a constant (independent of  $E, k$  and  $s$ ).

*Proof.* We again consider the preimage of  $V_E^{n-k}(s)$  in  $X$  under the map  $\pi$ . We need to estimate the volume of each horospherical piece

$$\mathcal{H}_{i\Theta}(s) = \bigcap_{\alpha \in \Theta} \{\tau_\alpha^{-1} \tilde{h}_{i\alpha} = -s\} \cap \mathcal{S}_i(s), \quad i \in E.$$

It clearly suffices to carry out the estimates for  $i = 1$ ; note that  $q_1 = e$ . For the horocyclic coordinate map  $\mu: Y \rightarrow X$  and the canonical projection

$\pi_A : Y \rightarrow A$  we set  $A_\Theta(s) := \pi_A \circ \mu^{-1}(\mathcal{H}_{1\Theta}(s)) \subset A$ . The set  $A_\Theta(s)$  is contained in an “affine” subspace of  $A$  of the form  $a_1 a_*(s) A^{q-k}$  where  $a_1 a_*(s) \in A$  and  $A^{q-k}$  is a  $q-k$ -dimensional subgroup of  $A$  (see Sections 3 and 4 of [L2]). We denote the restriction of  $dv_A$  to  $A^{q-k}$  by  $dv_{A^{q-k}}$ ; for  $k = q$  we have  $A^0 = e$  and we set  $dv_{A^0} \equiv 1$ . By Lemma 3.3 we have (for  $k$  equal to the number of elements of  $\Theta$ )

$$\text{Vol}(V_E^{n-k}(s)) \prec \int_{\mu^{-1}(\mathcal{H}_{1\Theta}(s))} \rho(a)^{-1} dv_U \wedge dv_Z \wedge dv_{A^{q-k}}.$$

Since the horospherical piece  $\mathcal{H}_{1\Theta}(s)$  is part of a Siegel set  $\mathcal{S}_{\omega, \tau}$  with  $\omega$  relatively compact (and hence of finite volume) in  $UM$  we get

$$\begin{aligned} \int_{\mu^{-1}(\mathcal{H}_{1\Theta}(s))} \rho(a)^{-1} dv_U \wedge dv_Z \wedge dv_{A^{q-k}} &\prec \\ &\prec \int_{\omega} dv_U \wedge dv_Z \int_{A_\Theta(s)} \rho(a)^{-1} dv_{A^{q-k}} \prec \int_{A_\Theta(s)} \rho(a)^{-1} dv_{A^{q-k}}. \end{aligned}$$

Also by definition of a Siegel set we have  $\alpha(a) \geq \tau \succ 1$  for all  $\alpha \in \Delta$ . Moreover, the computations in the proof of Lemma 4.1 (and Lemma 3.5) in [L2] show that for all  $\alpha \in \Theta$  one has  $\alpha(a_1 a_*(s)) \succ e^{\mu_\alpha s}$  with  $\mu_\alpha > 0$ . Hence, as  $\Theta \subset \Delta$  is not empty and since  $\rho = \sum_{\alpha \in \Delta} c_\alpha \alpha$  ( $c_\alpha > 0$ ), there is a uniform constant  $c > 0$  such that  $\rho(a)^{-1} \prec e^{-cs}$  for all  $a \in A_\Theta(s)$ . As noted above the set  $A_\Theta(s)$  is contained in a  $(q-k)$ -dimensional affine cone in  $A$ . It is similar (in the sense of Euclidean geometry) to  $A_\Theta(0)$  with similarity factor  $s$  (see the proof of Lemma 4.1 in [L2]). Hence we eventually get  $\int_{A_\Theta(s)} dv_{A^{q-k}} \prec s^{q-k}$  and the Lemma follows.  $\square$

#### 4. A NEW PROOF OF THE GAUSS-BONNET FORMULA

In this section we present a new simplified proof of the Gauss-Bonnet theorem for higher rank locally symmetric spaces.

**THEOREM 4.1.** *Let  $X$  be a Riemannian symmetric space of noncompact type and  $\mathbb{R}$ -rank  $\geq 2$  and let  $\Gamma$  be an irreducible, torsion-free (non-uniform) lattice in the group of isometries of  $X$ . Then for the locally symmetric space  $V = \Gamma \backslash X$  the Gauss-Bonnet formula holds:*

$$\chi(V) = \int_V \Psi dv.$$