

**Zeitschrift:** L'Enseignement Mathématique  
**Herausgeber:** Commission Internationale de l'Enseignement Mathématique  
**Band:** 42 (1996)  
**Heft:** 3-4: L'ENSEIGNEMENT MATHÉMATIQUE

**Artikel:** ON THE GAUSS-BONNET FORMULA FOR LOCALLY SYMMETRIC SPACES OF NONCOMPACT TYPE  
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**Kapitel:** 2.2. An exhaustion by polyhedra  
**DOI:** <https://doi.org/10.5169/seals-87876>

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standard geometric  $q-1$  simplex ( $q = \mathbb{Q}$ -rank of  $\mathbf{G}$ ). If  $\Delta = \{\alpha_1, \dots, \alpha_q\}$  and  $\Delta - \Theta = \{\alpha_{i_1}, \dots, \alpha_{i_s}\}$  with  $1 \leq i_1 < \dots < i_s \leq q$ , we define the boundary simplex  $\Delta(\Theta)$  of  $\Delta$  as  $\Delta(\Theta) := [e_{i_1}, \dots, e_{i_s}]$ . Let  $\mathbf{P}$  be a *minimal* parabolic  $\mathbb{Q}$ -subgroup of  $\mathbf{G}$  and let the set  $\Gamma \backslash \mathbf{G}(\mathbb{Q})/\mathbf{P}(\mathbb{Q})$  be represented by  $\{q_1, \dots, q_m\}$  (see Proposition 2.1). We take  $m$  copies  $\Delta^j = [e_1^j, \dots, e_q^j]$  of  $\Delta$  with faces  $\Delta^j(\Theta)$  corresponding to  $\Theta$ . The corresponding homeomorphisms  $\Delta \simeq \Delta^j$  are denoted by  $\varphi_j$ . The simplicial complex  $\Gamma \backslash |\mathcal{T}|$ , which provides a geometric realization of the quotient of the Tits building of  $\mathbf{G}$  modulo  $\Gamma$ , is constructed from the simplices  $\Delta^1, \dots, \Delta^m$  through the following *incidence relations*:

Two simplices  $\Delta^j$  and  $\Delta^l$  are pasted together along the faces  $\Delta^j(\Theta)$  and  $\Delta^l(\Theta)$  by the homeomorphism  $\varphi_j \circ \varphi_l^{-1} |_{\Delta^l(\Theta)}$  if and only if

$$\Gamma q_j \mathbf{P}_\Theta(\mathbb{Q}) = \Gamma q_l \mathbf{P}_\Theta(\mathbb{Q}).$$

We remark that the points of  $\Gamma \backslash |\mathcal{T}|$  are in one-to-one correspondence with equivalence classes of geodesic rays in the locally symmetric space  $V = \Gamma \backslash X$  (see [Hat], [L1] and [JM]).

## 2.2. AN EXHAUSTION BY POLYHEDRA

We index the “edges” of the Weyl chamber  $\overline{\mathfrak{a}^+}$  (or equivalently of  $\overline{A^+} \cdot x_0$ ) by *simple*  $\mathbb{Q}$ -roots. More precisely, the edges of  $\overline{A^+} \cdot x_0$  are given by geodesic rays  $c_\alpha(t) = \exp(tH_\alpha) \cdot x_0$  where  $H_\alpha \in \overline{\mathfrak{a}^+}$ ,  $\|H_\alpha\| = 1$  and  $\beta(H_\alpha) = 0$  for  $\beta \neq \alpha$  ( $\alpha, \beta \in \Delta$ ). We further write  $c_{k\alpha}$  for the edges  $q_k a_0 c_\alpha$  of the chambers  $q_k \mathcal{C}$  in the fundamental set  $\Omega$  (see Section 2.1 for the notation). If a geodesic ray  $c$  represents a point  $z \in \partial_\infty X$  we write  $z = c(\infty)$ . The group  $G$  act naturally on  $\partial_\infty X$  through  $g \cdot c(\infty) = (g \cdot c)(\infty)$ . For every  $\alpha \in \Delta$  the isotropy group of  $c_\alpha(\infty)$  under that action coincides with the (maximal) parabolic subgroups  $P_{\Delta - \{\alpha\}}$  introduced above (see [L2] Lemma 1.2).

To a geodesic ray  $c: [0, \infty) \rightarrow X$  (parametrized by arc-length) which represents a point  $z$  in the ideal boundary  $\partial_\infty X$  of  $X$  is associated a *Busemann function on  $X$  at  $z$*  given by

$$h_z: X \rightarrow \mathbb{R} \quad ; \quad h_z(x) = \lim_{t \rightarrow \infty} [d(x, c(t)) - t].$$

The level sets of a Busemann function are *horospheres*, which foliate the symmetric space. We denote the Busemann functions which correspond to the rays  $c_{k\alpha}$  by  $h_{k\alpha}$ . Note that  $h_{k\alpha}(c_{k\alpha}(t))$  tends to  $-\infty$  if the arc-length  $t$  of the geodesic  $c_{k\alpha}$  tends to  $+\infty$ .

In contrast to an exact fundamental domain there are not only points on the boundary of a fundamental set  $\Omega$  but possibly also interior points which are

identified under the action of  $\Gamma$ . However, there is only a finite set of isometries  $\gamma \in \Gamma$  with  $\gamma\Omega \cap \Omega \neq \emptyset$ . Furthermore it suffices to look at the (finite) set  $\mathcal{D}$  of those  $\gamma$  for which this intersection is not relatively compact in  $X$  (all other intersections are contained in some compact subset of  $\Omega$ ). It turns out that every  $\gamma \in \mathcal{D}$  has the crucial property that there are indices  $i, j$  such that  $q_j^{-1}\gamma q_i$  is parabolic i.e. fixes at least one point in the ideal boundary  $\partial_\infty X$  (see [L2] Proposition 2.2). Then for every  $\gamma \in \mathcal{D}$  there are indices  $i, j, \alpha$  such the family of horospheres of the form  $h_{i\alpha}^{-1}(s), s \in \mathbb{R}$ , is mapped isometrically to the family  $h_{j\alpha}^{-1}(s), s \in \mathbb{R}$  (see [L2] Lemma 3.2). These identifications correspond to the incidence relations described above in the construction of the simplicial complex  $\Gamma \backslash |\mathcal{T}|$ . (To see this one has to use the fact that the Siegel set at infinity  $\partial_\infty(q_j\mathcal{S})$  is canonically isomorphic to  $\Delta^j = [e_1^j, \dots, e_q^j]$ .) The main technical step is then to renormalize the Busemann functions as  $\tilde{h}_{i\alpha} = h_{i\alpha} - s_{ij}$  (for certain constants  $s_{ij}$ ) in such a way that each  $\gamma \in \mathcal{D}$  maps a horosphere of some given level, say  $\{\tilde{h}_{i\alpha} = s\}$ , to another one,  $\{\tilde{h}_{j\alpha} = s\}$ , of the *same* level  $s$  (see [L2] Lemma 3.4). This fact finally allows us to truncate the constituents  $q_i\mathcal{S}$  of the fundamental set  $\Omega$  by removing the open horoballs  $\mathcal{B}_{i\alpha}(s) := \{\tilde{h}_{i\alpha} < -\tau_\alpha s\}$  (for certain constants  $\tau_\alpha$  and for  $s > 0$  sufficiently large). The above construction guarantees that the truncated fundamental set  $\Omega(s) := \bigcup_{i=1}^m q_i\mathcal{S}(s)$  of  $\Omega$  is relatively compact in  $X$  and invariant under the (restricted) action of  $\Gamma$ . Moreover for  $s$  sufficiently large the  $\Gamma$ -invariant “core”  $X(s) := \Gamma \cdot \Omega(s)$  can be written as the complement in  $X$  of a union of (countably many) open horoballs:  $X(s) = X - \Gamma \cdot \bigcup_{i=1}^m \bigcup_{\alpha \in \Delta} \mathcal{B}_{i\alpha}(s)$  (see [L3] Theorem 3.6). These horoballs are disjoint if and only if  $\Gamma$  is an arithmetic subgroup of a  $\mathbb{Q}$ -rank 1 group. The projection  $\pi : X \rightarrow V$  maps  $X(s)$  to a compact submanifold with corners  $V(s)$  of  $V$  whose fundamental group is isomorphic to  $\Gamma$ . The “centers” of the projected horoballs in  $\partial_\infty V$  are in bijection with the vertices of  $\Gamma \backslash |\mathcal{T}|$ . The exhaustion function  $h$  is eventually defined in such a way that its level sets coincide with the boundaries  $\partial V(s)$ . We summarize the result in the following proposition (see [L2] Theorem 4.2).

**PROPOSITION 2.2.** *Let  $X$  be a Riemannian symmetric space of noncompact type and  $\mathbb{R}$ -rank  $\geq 2$  and let  $\Gamma$  be an irreducible, torsion-free, non-uniform lattice in the group of isometries of  $X$ . On the locally symmetric space  $V = \Gamma \backslash X$  there exists a piecewise real analytic exhaustion function  $h : V \rightarrow [0, \infty)$  such that, for each  $s \geq 0$ , the sublevel set  $V(s) := \{h \leq s\}$  is a Riemannian polyhedron in  $V$ . Moreover the level sets  $\{h = s\} = \partial V(s)$  consist of projections of pieces of horospheres in  $X$ .*

Each polyhedron  $V(s)$  is homotopically equivalent to  $V$ . More precisely we have

**PROPOSITION 2.3.** *For every sufficiently large  $s$  the locally symmetric space  $V$  is homeomorphic to the interior of the polyhedron  $V(s)$  in  $V$ , and  $V(s)$  is a strong deformation retract of  $V$ .*

For the proof see [L3], Theorems 5.2 and 5.5.

### 3. ESTIMATES FOR THE BOUNDARY SUBPOLYHEDRA

We wish to apply Proposition 1.1 to the polyhedra  $V(s)$  in the above exhaustion and then take the limit for  $s \rightarrow \infty$ . To that end we need estimates for the second fundamental forms and the volumes of the (lower dimensional) boundary polyhedra.

For each Siegel set  $\mathcal{S}_i := q_i \mathcal{S}$  which is part of the fundamental set  $\Omega$  we have its truncated part

$$\mathcal{S}_i(s) := \mathcal{S}_i - \bigcup_{\alpha \in \Delta} (\mathcal{B}_{i\alpha}(s) \cap \mathcal{S}_i).$$

The top dimensional boundary faces of  $\mathcal{S}_i(s)$  in  $\mathcal{S}_i$  (resp. of  $\Omega(s)$  in  $\Omega$ ) are subsets of horospheres:

$$\mathcal{H}_{i\alpha}(s) := \{\tau_\alpha^{-1} \tilde{h}_{i\alpha} = -s\} \cap \mathcal{S}_i(s), \quad \alpha \in \Delta.$$

The “horospherical” pieces  $\mathcal{H}_{i\alpha}(s)$  together with their  $\Gamma$ -translates form the boundary of the manifold with corners  $X(s)$  in  $X$ . For any nonempty subset  $\Theta$  of  $\Delta$  we set

$$\mathcal{H}_{i\Theta}(s) := \bigcap_{\alpha \in \Theta} \mathcal{H}_{i\alpha}(s) \subset \mathcal{S}_i(s).$$

The various boundary subpolyhedra of  $V(s)$  are then unions of projections of the pieces  $\mathcal{H}_{i\Theta}(s)$  under the canonical projection  $\pi : X \rightarrow V$ . More precisely, as explained in Section 2, for any subset  $\Theta \subset \Delta$ , we have the equivalence relation on the set  $I = \{1, \dots, m\}$

$$j \sim_\Theta l \quad \text{if and only if} \quad \Gamma q_j P_\Theta = \Gamma q_l P_\Theta$$

(the  $q_i$  are as in Proposition 2.1). This relation  $\sim_\Theta$  induces a partition,  $I(\Theta)$ , of the set  $I$  whose components will be denoted by  $E$ . Let  $n = \dim X = \dim V$ , let  $k$  be the cardinality of  $\Theta$  and let  $E \in I(\Theta)$ . Then  $V_E^{n-k}(s) := \pi(\bigcup_{i \in E} \mathcal{H}_{i\Theta}(s))$  is a  $(n - k)$ -dimensional boundary polyhedron of  $V(s)$ ; and moreover, any boundary polyhedron arises in this way (see [L3] §4).