Zeitschrift:	L'Enseignement Mathématique
Herausgeber:	Commission Internationale de l'Enseignement Mathématique
Band:	42 (1996)
Heft:	3-4: L'ENSEIGNEMENT MATHÉMATIQUE
Artikel:	ON THE GAUSS-BONNET FORMULA FOR LOCALLY SYMMETRIC SPACES OF NONCOMPACT TYPE
Autor:	Leuzinger, Enrico
Kapitel:	2.2. An exhaustion by polyhedra
DOI:	https://doi.org/10.5169/seals-87876

Nutzungsbedingungen

Die ETH-Bibliothek ist die Anbieterin der digitalisierten Zeitschriften auf E-Periodica. Sie besitzt keine Urheberrechte an den Zeitschriften und ist nicht verantwortlich für deren Inhalte. Die Rechte liegen in der Regel bei den Herausgebern beziehungsweise den externen Rechteinhabern. Das Veröffentlichen von Bildern in Print- und Online-Publikationen sowie auf Social Media-Kanälen oder Webseiten ist nur mit vorheriger Genehmigung der Rechteinhaber erlaubt. <u>Mehr erfahren</u>

Conditions d'utilisation

L'ETH Library est le fournisseur des revues numérisées. Elle ne détient aucun droit d'auteur sur les revues et n'est pas responsable de leur contenu. En règle générale, les droits sont détenus par les éditeurs ou les détenteurs de droits externes. La reproduction d'images dans des publications imprimées ou en ligne ainsi que sur des canaux de médias sociaux ou des sites web n'est autorisée qu'avec l'accord préalable des détenteurs des droits. <u>En savoir plus</u>

Terms of use

The ETH Library is the provider of the digitised journals. It does not own any copyrights to the journals and is not responsible for their content. The rights usually lie with the publishers or the external rights holders. Publishing images in print and online publications, as well as on social media channels or websites, is only permitted with the prior consent of the rights holders. <u>Find out more</u>

Download PDF: 19.08.2025

ETH-Bibliothek Zürich, E-Periodica, https://www.e-periodica.ch

standard geometric q-1 simplex ($q = \mathbb{Q}$ -rank of **G**). If $\Delta = \{\alpha_1, \ldots, \alpha_q\}$ and $\Delta - \Theta = \{\alpha_{i_1}, \ldots, \alpha_{i_s}\}$ with $1 \leq i_1 < \ldots < i_s \leq q$, we define the boundary simplex $\Delta(\Theta)$ of Δ as $\Delta(\Theta) := [e_{i_1}, \ldots, e_{i_s}]$. Let **P** be a minimal parabolic \mathbb{Q} -subgroup of **G** and let the set $\Gamma \setminus \mathbf{G}(\mathbb{Q})/\mathbf{P}(\mathbb{Q})$ be represented by $\{q_1, \ldots, q_m\}$ (see Proposition 2.1). We take *m* copies $\Delta^j = [e_1^j, \ldots, e_q^j]$ of Δ with faces $\Delta^j(\Theta)$ corresponding to Θ . The corresponding homeomorphisms $\Delta \simeq \Delta^j$ are denoted by φ_j . The simplicial complex $\Gamma \setminus |\mathcal{T}|$, which provides a geometric realization of the quotient of the Tits building of **G** modulo Γ , is constructed from the simplices $\Delta^1, \ldots, \Delta^m$ through the following incidence relations:

Two simplices \triangle^j and \triangle^l are pasted together along the faces $\triangle^j(\Theta)$ and $\triangle^l(\Theta)$ by the homeomorphism $\varphi_j \circ \varphi_l^{-1} \mid_{\triangle^l(\Theta)}$ if and only if

$$\Gamma q_{l} \mathbf{P}_{\Theta}(\mathbb{Q}) = \Gamma q_{l} \mathbf{P}_{\Theta}(\mathbb{Q}) \,.$$

We remark that the points of $\Gamma \setminus |\mathcal{T}|$ are in one-to-one correspondence with equivalence classes of geodesic rays in the locally symmetric space $V = \Gamma \setminus X$ (see [Hat], [L1] and [JM]).

2.2. AN EXHAUSTION BY POLYHEDRA

We index the "edges" of the Weyl chamber $\overline{\mathfrak{a}^+}$ (or equivalently of $\overline{A^+} \cdot x_0$) by simple \mathbb{Q} -roots. More precisely, the edges of $\overline{A^+} \cdot x_0$ are given by geodesic rays $c_{\alpha}(t) = \exp(tH_{\alpha}) \cdot x_0$ where $H_{\alpha} \in \overline{\mathfrak{a}^+}$, $||H_{\alpha}|| = 1$ and $\beta(H_{\alpha}) = 0$ for $\beta \neq \alpha$ ($\alpha, \beta \in \Delta$). We further write $c_{k\alpha}$ for the edges $q_k a_0 c_{\alpha}$ of the chambers $q_k C$ in the fundamental set Ω (see Section 2.1 for the notation). If a geodesic ray c represents a point $z \in \partial_{\infty} X$ we write $z = c(\infty)$. The group G act naturally on $\partial_{\infty} X$ through $g \cdot c(\infty) = (g \cdot c)(\infty)$. For every $\alpha \in \Delta$ the isotropy group of $c_{\alpha}(\infty)$ under that action coincides with the (maximal) parabolic subgroups $P_{\Delta - \{\alpha\}}$ introduced above (see [L2] Lemma 1.2).

To a geodesic ray $c: [0, \infty) \longrightarrow X$ (parametrized by arc-length) which represents a point z in the ideal boundary $\partial_{\infty} X$ of X is associated a *Busemann* function on X at z given by

$$h_z: X \longrightarrow \mathbb{R}$$
; $h_z(x) = \lim_{t \to \infty} \left[d(x, c(t)) - t \right]$.

The level sets of a Busemann function are *horospheres*, which foliate the symmetric space. We denote the Busemann functions which correspond to the rays $c_{k\alpha}$ by $h_{k\alpha}$. Note that $h_{k\alpha}(c_{k\alpha}(t))$ tends to $-\infty$ if the arc-length t of the geodesic $c_{k\alpha}$ tends to $+\infty$.

In contrast to an exact fundamental domain there are not only points on the boundary of a fundamental set Ω but possibly also interior points which are

identified under the action of Γ . However, there is only a finite set of isometries $\gamma \in \Gamma$ with $\gamma \Omega \cap \Omega \neq \emptyset$. Furthermore it suffices to look at the (finite) set \mathcal{D} of those γ for which this intersection is not relatively compact in X (all other intersections are contained in some compact subset of Ω). It turns out that every $\gamma \in \mathcal{D}$ has the crucial property that there are indices *i*, *j* such that $q_i^{-1}\gamma q_i$ is parabolic i.e. fixes at least one point in the ideal boundary $\partial_{\infty} X$ (see [L2] Proposition 2.2). Then for every $\gamma \in \mathcal{D}$ there are indices i, j, α such the family of horospheres of the form $h_{i\alpha}^{-1}(s), s \in \mathbb{R}$, is mapped isometrically to the family $h_{j\alpha}^{-1}(s), s \in \mathbb{R}$ (see [L2] Lemma 3.2). These identifications correspond to the incidence relations described above in the construction of the simplicial complex $\Gamma \setminus |\mathcal{T}|$. (To see this one has to use the fact that the Siegel set at infinity $\partial_{\infty}(q_i S)$ is canonically isomorphic to $\Delta^j = [e_1^j, \ldots, e_q^j]$.) The main technical step is then to renormalize the Busemann functions as $\tilde{h}_{i\alpha} = h_{i\alpha} - s_{ij}$ (for certain constants s_{ij}) in such a way that each $\gamma \in \mathcal{D}$ maps a horosphere of some given level, say $\{\tilde{h}_{i\alpha} = s\}$, to another one, $\{\tilde{h}_{j\alpha} = s\}$, of the same level s (see [L2] Lemma 3.4). This fact finally allows us to truncate the constituents $q_i S$ of the fundamental set Ω by removing the open horoballs $\mathcal{B}_{i\alpha}(s) := \{\tilde{h}_{i\alpha} < -\tau_{\alpha}s\}$ (for certain constants au_{lpha} and for s>0 sufficiently large). The above construction guarantees that the truncated fundamental set $\Omega(s) := \bigcup_{i=1}^{m} q_i \mathcal{S}(s)$ of Ω is relatively compact in X and invariant under the (restricted) action of Γ . Moreover for s sufficiently large the Γ -invariant "core" $X(s) := \Gamma \cdot \Omega(s)$ can be written as the complement in X of a union of (countably many) open horoballs: $X(s) = X - \Gamma \cdot \bigcup_{i=1}^{m} \bigcup_{\alpha \in \Delta} \mathcal{B}_{i\alpha}(s)$ (see [L3] Theorem 3.6). These horoballs are disjoint if and only if Γ is an arithmetic subgroup of a \mathbb{Q} -rank 1 group. The projection $\pi: X \longrightarrow V$ maps X(s) to a compact submanifold with corners V(s) of V whose fundamental group is isomorphic to Γ . The "centers" of the projected horoballs in $\partial_{\infty}V$ are in bijection with the vertices of $\Gamma \setminus |\mathcal{T}|$. The exhaustion function h is eventually defined in such a way that its level sets coincide with the boundaries $\partial V(s)$. We summarize the result in the following proposition (see [L2] Theorem 4.2).

PROPOSITION 2.2. Let X be a Riemannian symmetric space of noncompact type and \mathbb{R} -rank ≥ 2 and let Γ be an irreducible, torsion-free, nonuniform lattice in the group of isometries of X. On the locally symmetric space $V = \Gamma \setminus X$ there exists a piecewise real analytic exhaustion function $h: V \longrightarrow [0, \infty)$ such that, for each $s \geq 0$, the sublevel set $V(s) := \{h \leq s\}$ is a Riemannian polyhedron in V. Moreover the level sets $\{h = s\} = \partial V(s)$ consist of projections of pieces of horospheres in X. Each polyhedron V(s) is homotopically equivalent to V. More precisely we have

PROPOSITION 2.3. For every sufficiently large s the locally symmetric space V is homeomorphic to the interior of the polyhedron V(s) in V, and V(s) is a strong deformation retract of V.

For the proof see [L3], Theorems 5.2 and 5.5.

3. ESTIMATES FOR THE BOUNDARY SUBPOLYHEDRA

We wish to apply Proposition 1.1 to the polyhedra V(s) in the above exhaustion and then take the limit for $s \to \infty$. To that end we need estimates for the second fundamental forms and the volumes of the (lower dimensional) boundary polyhedra.

For each Siegel set $S_i := q_i S$ which is part of the fundamental set Ω we have its truncated part

$$\mathcal{S}_i(s) := \mathcal{S}_i - \bigcup_{\alpha \in \Delta} (\mathcal{B}_{i\alpha}(s) \cap \mathcal{S}_i).$$

The top dimensional boundary faces of $S_i(s)$ in S_i (resp. of $\Omega(s)$ in Ω) are subsets of horospheres :

$$\mathcal{H}_{i\alpha}(s) := \{\tau_{\alpha}^{-1}\tilde{h}_{i\alpha} = -s\} \cap \mathcal{S}_{i}(s) , \quad \alpha \in \Delta .$$

The "horospherical" pieces $\mathcal{H}_{i\alpha}(s)$ together with their Γ -translates form the boundary of the manifold with corners X(s) in X. For any nonempty subset Θ of Δ we set

$$\mathcal{H}_{i\Theta}(s) := \bigcap_{\alpha \in \Theta} \mathcal{H}_{i\alpha}(s) \subset \mathcal{S}_i(s).$$

The various boundary subpolyhedra of V(s) are then unions of projections of the pieces $\mathcal{H}_{i\Theta}(s)$ under the canonical projection $\pi: X \to V$. More precisely, as explained in Section 2, for any subset $\Theta \subset \Delta$, we have the equivalence relation on the set $I = \{1, \ldots, m\}$

$$j \sim_{\Theta} l$$
 if and only if $\Gamma q_j P_{\Theta} = \Gamma q_l P_{\Theta}$

(the q_i are as in Proposition 2.1). This relation \sim_{Θ} induces a partition, $I(\Theta)$, of the set I whose components will be denoted by E. Let $n = \dim X = \dim V$, let k be the cardinality of Θ and let $E \in I(\Theta)$. Then $V_E^{n-k}(s) := \pi \left(\bigcup_{i \in E} \mathcal{H}_{i\Theta}(s) \right)$ is a (n-k)-dimensional boundary polyhedron of V(s); and moreover, any boundary polyhedron arises in this way (see [L3] §4).