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#### E. LEUZINGER

# 2. AN EXHAUSTION OF LOCALLY SYMMETRIC SPACES

Let X be a Riemannian symmetric space of noncompact type and rank  $\geq 2$ and let  $\Gamma$  be a non-uniform, torsion-free lattice in the group of isometries of X. In this section we briefly describe the basic features of an exhaustion of the locally symmetric space  $V = \Gamma \setminus X$  by Riemannian polyhedra, which was previously constructed in [L2].

The idea is to work with a *fundamental set*  $\Omega \subset X$  for the discrete (arithmetic) group  $\Gamma$ . Such "coarse" fundamental domains are provided by *reduction theory*; they are finite unions of translates of so-called Siegel sets. We begin with reviewing some facts about linear algebraic groups and set up the notation. Roughly speaking, the lattice  $\Gamma$  determines a "Q-structure" on the real Lie group of isometries of X such that  $\Gamma$  is given by integer matrices. The symmetric space X in turn inherits canonical parametrizations adopted to this structure (generalized horocyclic coordinates). Siegel sets are then defined with respect to such parametrizations.

# 2.1. REDUCTION THEORY AND GEOMETRY AT INFINITY

We denote by *G* the identity component of the group of isometries of *X*; it is a connected, semisimple Lie group with trivial center. We shall always assume in the following that the non-uniform lattice  $\Gamma$  is *irreducible* (see [R2] 5.20). Then, by the arithmeticity theorem of Margulis, there is a connected semisimple linear algebraic group **G** defined over  $\mathbb{Q}$ ,  $\mathbb{Q}$ -embedded in a general linear group  $\mathbf{GL}(N, \mathbb{C})$ , and a Lie group isomorphism  $p: G \longrightarrow \mathbf{G}(\mathbb{R})^0$  such that  $p(\Gamma)$  is *arithmetic*, i.e.  $p(\Gamma) \subset \mathbf{G}(\mathbb{Q}) \subset \mathbf{GL}(N, \mathbb{C})$  is commensurable with the group  $\mathbf{G}(\mathbb{Z}) = \mathbf{G} \cap \mathbf{GL}(N, \mathbb{Z})$  (see [Z] 3.1.6 and 6.1.10). The symmetric space *X* can be recovered as the manifold of maximal compact subgroups of the identity component of the group  $\mathbf{G}(\mathbb{R}) = \mathbf{G} \cap \mathbf{GL}(N, \mathbb{R})$  of  $\mathbb{R}$ -rational points of **G**. For simplicity we will always identify *G* with  $\mathbf{G}(\mathbb{R})^0$  and  $\Gamma$ with  $p(\Gamma)$ .

Let **S** (resp. **T**) be a maximal  $\mathbb{Q}$ -split (resp.  $\mathbb{R}$ -split) algebraic torus of **G**, i.e. a subgroup of **G** which is isomorphic over  $\mathbb{Q}$  (resp.  $\mathbb{R}$ ) to the direct product of q (resp.  $r \ge q$ ) copies of  $\mathbb{C}^*$ . All such tori are conjugate under  $\mathbf{G}(\mathbb{Q}) = \mathbf{G} \cap \mathbf{GL}(N, \mathbb{Q})$  (resp.  $\mathbf{G}(\mathbb{R})$ ) and their common dimension q (resp. r) is called the  $\mathbb{Q}$ -rank (resp.  $\mathbb{R}$ -rank) of **G**. The identity component of  $\mathbf{S}(\mathbb{R})$  (resp.  $\mathbf{T}(\mathbb{R})$ ) will be denoted by A (resp.  $A_0$ ), the corresponding Lie algebras by  $\mathfrak{a}$  (resp.  $\mathfrak{a}_0$ ). The  $\mathbb{R}$ -rank of **G** coincides with the rank of the symmetric space X, i.e. the maximal dimension of totally geodesic flat subspaces. The choice of a maximal compact subgroup K of G is equivalent to the choice of a base point  $x_0$  of X. We can choose K with Lie algebra & so that under the corresponding Cartan decomposition  $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$  of the Lie algebra  $\mathfrak{g}$  of G we have  $\mathfrak{a} \subseteq \mathfrak{a}_0 \subset \mathfrak{p} \cong T_{x_0}X$ . Here  $a_0$  is maximal abelian in p, i.e. the tangent space at  $x_0$  of the (maximal  $\mathbb{R}$ -) flat  $A_0 \cdot x_0$  in X. The pair of Lie algebras  $(\mathfrak{g}, \mathfrak{a}_0)$  gives rise to the root system  ${}_{\mathbb{R}}\Phi$  of the symmetric space. Similarly there is a system of Q-roots  $_{\mathbb{Q}}\Phi$  associated to the pair (g, a) (see [B3] §21). It is always possible to choose orderings of  ${}_{\mathbb{O}}\Phi$  and  ${}_{\mathbb{R}}\Phi$  such that the restrictions of simple  $\mathbb{R}$ -roots of  $\mathbb{R}\Phi$  to a are either simple  $\mathbb{Q}$ -roots of  $\mathbb{Q}\Phi$ , i.e. the elements of a basis  $\Delta = {}_{\mathbb{Q}}\Delta$  of  ${}_{\mathbb{Q}}\Phi$ , or zero (see [BT] 6.8). The basis  $\mathbb{R}^{\Delta}$  defines a closed  $\mathbb{R}$ -Weyl chamber  $\overline{\mathfrak{a}_0^+}$  in  $\mathfrak{a}_0$  and  $\Delta$  then determines a closed  $\mathbb{Q}$ -Weyl chamber  $\overline{\mathfrak{a}^+} := \{H \in \mathfrak{a} \mid \alpha(H) \ge 0, \text{ for all } \alpha \in \Delta\}$  in  $\mathfrak{a}$ . We set  $\overline{A^+} = \exp \overline{\mathfrak{a}^+}$  (resp.  $\overline{A_0^+} = \exp \overline{\mathfrak{a}_0^+}$ ). A Q-Weyl chamber in X is a translate of the basic chamber  $\overline{A^+} \cdot x_0 \subseteq \overline{A_0^+} \cdot x_0$ . The elements of  $\Delta$  are differentials of characters (defined over  $\mathbb{Q}$ ) of the maximal  $\mathbb{Q}$ -split torus S. It is convenient to identify the elements of  $\Delta$  also with such characters. When restricted to A their values are denoted by  $\alpha(a)$  ( $a \in A, \alpha \in \Delta$ ). Notice that  $\overline{A^+} = \{ a \in A \mid \alpha(a) \ge 1 \text{ for all } \alpha \in \Delta \}.$ 

A closed subgroup **P** of **G** defined over  $\mathbb{O}$  is a *parabolic*  $\mathbb{O}$ -subgroup if G/P is a projective variety (see [B3] §11). A parabolic  $\mathbb{O}$ -subgroup P of  $G = \mathbf{G}(\mathbb{R})^0$  is by definition the intersection of G with a parabolic  $\mathbb{Q}$ -subgroup of **G** (see [BS]). The conjugacy classes under  $\mathbf{G}(\mathbb{Q})$  of parabolic  $\mathbb{Q}$ -subgroups are in one-to-one correspondence with the subsets  $\Theta$  of the (chosen) set  $\Delta$ of simple  $\mathbb{Q}$ -roots; they are represented by the *standard parabolic*  $\mathbb{Q}$ -subgroups  $\mathbf{P}_{\Theta}$  of **G** (see [B3] §21.11). The corresponding standard parabolic Q-subgroups of G are denoted by  $P_{\Theta}$ . The minimal parabolic subgroup  $P = P_{\varnothing}$  has a decomposition P = UMA, where U is unipotent and M is reductive; A centralizes M and normalizes U (see [B1]). This yields a (generalized) Iwasawa decomposition for G, i.e.  $G = P \cdot K = UMAK$ , which implies that P acts transitively on the symmetric space X. The intersection of the maximal compact subgroup K of G with M is maximal compact in M and the quotient  $Z = M/(K \cap M)$  is (in general) the Riemannian product of a symmetric space of noncompact type by a (flat) Euclidean space. Let  $\tau: M \longrightarrow Z$  be the natural projection. Then the "horocyclic coordinate map"

 $\mu: Y = U \times Z \times A \longmapsto X \quad ; \quad (u, \tau(m), a) \longmapsto uma \cdot x_0$ 

is an isomorphism of analytic manifolds (see [BS] or [B2]).

A generalized Siegel set  $S = S_{\omega,\tau}$  in X (relative to the Q-Weyl chamber  $\overline{A^+} \cdot x_0$ ) is a subset of X of the form  $S_{\omega,\tau} = \omega A_\tau \cdot x_0$  where  $\omega$  is relatively compact in UM and, for  $\tau > 0$ ,  $A_\tau = \{a \in A \mid \alpha(a) \ge \tau, \alpha \in \Delta\}$ . If we define  $a_0 \in A$  by  $\alpha(a_0) = \tau$  for all  $\alpha \in \Delta$ , then  $A_\tau = A_1 a_0 = \overline{A^+} a_0$  and  $\mathcal{C} = A_\tau \cdot x_0 \subset S$  is a (translate of a) Q-Weyl chamber in X. Siegel sets provide the building blocks for (approximate) fundamental domains for arithmetic groups. A subset  $\Omega \subset X$  is called a *fundamental set* for an arithmetic group  $\Gamma$  if the following two conditions hold

(i)  $X = \Gamma \cdot \Omega$ ;

(ii) for every  $q \in \mathbf{G}(\mathbb{Q})$  the set  $\{\gamma \in \Gamma \mid q\Omega \cap \gamma\Omega \neq \emptyset\}$  is finite.

The existence of fundamental sets is guaranteed by reduction theory for arithmetic groups (see [B1] §13 and §15).

PROPOSITION 2.1 (Borel, Harish-Chandra). Let **G** be a semisimple algebraic group defined over  $\mathbb{Q}$  with associated Riemannian symmetric space X = G/K. Let **P** be a minimal parabolic  $\mathbb{Q}$ -subgroup of **G** and let  $\Gamma$  be an arithmetic subgroup of  $\mathbf{G}(\mathbb{Q})$ . Then there exists a generalized Siegel set  $S = S_{\omega,\tau}$  (with respect to  $\overline{A^+} \cdot x_0$ ) such that, for a (fixed) set  $\{q_i \mid 1 \leq i \leq m\}$  of representatives of the finite set of double cosets  $\Gamma \setminus \mathbf{G}(\mathbb{Q})/\mathbf{P}(\mathbb{Q})$ , the union  $\Omega = \bigcup_{i=1}^m q_i \cdot S$  is a fundamental set (of finite volume) for  $\Gamma$  in X.

It will be useful in the sequel to dispose of geometric interpretations of the above algebraic concepts and assertions.

First recall that the symmetric space X, as a Riemannian manifold of nonpositive curvature, has an *ideal boundary at infinity*  $\partial_{\infty}X$ . The latter is defined as the set of equivalence classes of asymptotic geodesic rays (see [BGS]). In the same way one also defines the ideal boundary at infinity  $\partial_{\infty}V$  of  $V = \Gamma \setminus X$ . If  $\Gamma$  is an arithmetic lattice in a group **G** of  $\mathbb{Q}$ -rank q = 1, the boundary  $\partial_{\infty}V$  of the associated locally symmetric space consists of *m* points (corresponding to the cusps), where *m* is as in Proposition 2.1. For  $\mathbb{Q}$ -rank  $q \ge 2$  it turns out that  $\partial_{\infty}V$  is isomorphic to a finite simplicial complex  $\Gamma \setminus |\mathcal{T}|$ , a geometric realization of the Tits building of **G** modulo  $\Gamma$ (see [JM] and [L1]). We recall the construction of the latter.

Let  $\mathcal{P}$  be the set of all parabolic  $\mathbb{Q}$ -subgroups of  $\mathbf{G}$ . The conjugacy classes of elements of  $\mathcal{P}$  are in one-to-one correspondence with the subsets  $\Theta$  of the set  $\Delta$  of simple  $\mathbb{Q}$ -roots. Every conjugacy class has a standard representative denoted by  $\mathbf{P}_{\Theta}$ . One can show that the sets of double cosets  $\Gamma \setminus \mathbf{G}(\mathbb{Q}) / \mathbf{P}_{\Theta}(\mathbb{Q})$ are *finite* for all  $\Theta$  (see [B1], §15.6). Let  $\Delta = [e_1, \ldots, e_q] \subset \mathbb{R}^q$  denote a standard geometric q-1 simplex ( $q = \mathbb{Q}$ -rank of **G**). If  $\Delta = \{\alpha_1, \ldots, \alpha_q\}$  and  $\Delta - \Theta = \{\alpha_{i_1}, \ldots, \alpha_{i_s}\}$  with  $1 \leq i_1 < \ldots < i_s \leq q$ , we define the boundary simplex  $\Delta(\Theta)$  of  $\Delta$  as  $\Delta(\Theta) := [e_{i_1}, \ldots, e_{i_s}]$ . Let **P** be a minimal parabolic  $\mathbb{Q}$ -subgroup of **G** and let the set  $\Gamma \setminus \mathbf{G}(\mathbb{Q})/\mathbf{P}(\mathbb{Q})$  be represented by  $\{q_1, \ldots, q_m\}$  (see Proposition 2.1). We take *m* copies  $\Delta^j = [e_1^j, \ldots, e_q^j]$  of  $\Delta$  with faces  $\Delta^j(\Theta)$  corresponding to  $\Theta$ . The corresponding homeomorphisms  $\Delta \simeq \Delta^j$  are denoted by  $\varphi_j$ . The simplicial complex  $\Gamma \setminus |\mathcal{T}|$ , which provides a geometric realization of the quotient of the Tits building of **G** modulo  $\Gamma$ , is constructed from the simplices  $\Delta^1, \ldots, \Delta^m$  through the following incidence relations:

Two simplices  $\triangle^j$  and  $\triangle^l$  are pasted together along the faces  $\triangle^j(\Theta)$  and  $\triangle^l(\Theta)$  by the homeomorphism  $\varphi_j \circ \varphi_l^{-1} \mid_{\triangle^l(\Theta)}$  if and only if

$$\Gamma q_{l} \mathbf{P}_{\Theta}(\mathbb{Q}) = \Gamma q_{l} \mathbf{P}_{\Theta}(\mathbb{Q}) \,.$$

We remark that the points of  $\Gamma \setminus |\mathcal{T}|$  are in one-to-one correspondence with equivalence classes of geodesic rays in the locally symmetric space  $V = \Gamma \setminus X$  (see [Hat], [L1] and [JM]).

## 2.2. AN EXHAUSTION BY POLYHEDRA

We index the "edges" of the Weyl chamber  $\overline{\mathfrak{a}^+}$  (or equivalently of  $\overline{A^+} \cdot x_0$ ) by simple  $\mathbb{Q}$ -roots. More precisely, the edges of  $\overline{A^+} \cdot x_0$  are given by geodesic rays  $c_{\alpha}(t) = \exp(tH_{\alpha}) \cdot x_0$  where  $H_{\alpha} \in \overline{\mathfrak{a}^+}$ ,  $||H_{\alpha}|| = 1$  and  $\beta(H_{\alpha}) = 0$ for  $\beta \neq \alpha$  ( $\alpha, \beta \in \Delta$ ). We further write  $c_{k\alpha}$  for the edges  $q_k a_0 c_{\alpha}$  of the chambers  $q_k C$  in the fundamental set  $\Omega$  (see Section 2.1 for the notation). If a geodesic ray c represents a point  $z \in \partial_{\infty} X$  we write  $z = c(\infty)$ . The group G act naturally on  $\partial_{\infty} X$  through  $g \cdot c(\infty) = (g \cdot c)(\infty)$ . For every  $\alpha \in \Delta$ the isotropy group of  $c_{\alpha}(\infty)$  under that action coincides with the (maximal) parabolic subgroups  $P_{\Delta - \{\alpha\}}$  introduced above (see [L2] Lemma 1.2).

To a geodesic ray  $c: [0, \infty) \longrightarrow X$  (parametrized by arc-length) which represents a point z in the ideal boundary  $\partial_{\infty} X$  of X is associated a *Busemann* function on X at z given by

$$h_z: X \longrightarrow \mathbb{R}$$
;  $h_z(x) = \lim_{t \to \infty} \left[ d(x, c(t)) - t \right]$ .

The level sets of a Busemann function are *horospheres*, which foliate the symmetric space. We denote the Busemann functions which correspond to the rays  $c_{k\alpha}$  by  $h_{k\alpha}$ . Note that  $h_{k\alpha}(c_{k\alpha}(t))$  tends to  $-\infty$  if the arc-length t of the geodesic  $c_{k\alpha}$  tends to  $+\infty$ .

In contrast to an exact fundamental domain there are not only points on the boundary of a fundamental set  $\Omega$  but possibly also interior points which are