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## 2. AN EXHAUSTION OF LOCALLY SYMMETRIC SPACES

Let  $X$  be a Riemannian symmetric space of noncompact type and rank  $\geq 2$  and let  $\Gamma$  be a non-uniform, torsion-free lattice in the group of isometries of  $X$ . In this section we briefly describe the basic features of an exhaustion of the locally symmetric space  $V = \Gamma \backslash X$  by Riemannian polyhedra, which was previously constructed in [L2].

The idea is to work with a *fundamental set*  $\Omega \subset X$  for the discrete (arithmetic) group  $\Gamma$ . Such “coarse” fundamental domains are provided by *reduction theory*; they are finite unions of translates of so-called Siegel sets. We begin with reviewing some facts about linear algebraic groups and set up the notation. Roughly speaking, the lattice  $\Gamma$  determines a “ $\mathbb{Q}$ -structure” on the real Lie group of isometries of  $X$  such that  $\Gamma$  is given by integer matrices. The symmetric space  $X$  in turn inherits canonical parametrizations adopted to this structure (generalized horocyclic coordinates). Siegel sets are then defined with respect to such parametrizations.

## 2.1. REDUCTION THEORY AND GEOMETRY AT INFINITY

We denote by  $G$  the identity component of the group of isometries of  $X$ ; it is a connected, semisimple Lie group with trivial center. We shall always assume in the following that the non-uniform lattice  $\Gamma$  is *irreducible* (see [R2] 5.20). Then, by the arithmeticity theorem of Margulis, there is a connected semisimple linear algebraic group  $\mathbf{G}$  defined over  $\mathbb{Q}$ ,  $\mathbb{Q}$ -embedded in a general linear group  $\mathbf{GL}(N, \mathbb{C})$ , and a Lie group isomorphism  $p : G \rightarrow \mathbf{G}(\mathbb{R})^0$  such that  $p(\Gamma)$  is *arithmetic*, i.e.  $p(\Gamma) \subset \mathbf{G}(\mathbb{Q}) \subset \mathbf{GL}(N, \mathbb{C})$  is commensurable with the group  $\mathbf{G}(\mathbb{Z}) = \mathbf{G} \cap \mathbf{GL}(N, \mathbb{Z})$  (see [Z] 3.1.6 and 6.1.10). The symmetric space  $X$  can be recovered as the manifold of maximal compact subgroups of the identity component of the group  $\mathbf{G}(\mathbb{R}) = \mathbf{G} \cap \mathbf{GL}(N, \mathbb{R})$  of  $\mathbb{R}$ -rational points of  $\mathbf{G}$ . For simplicity we will always identify  $G$  with  $\mathbf{G}(\mathbb{R})^0$  and  $\Gamma$  with  $p(\Gamma)$ .

Let  $\mathbf{S}$  (resp.  $\mathbf{T}$ ) be a maximal  $\mathbb{Q}$ -split (resp.  $\mathbb{R}$ -split) algebraic torus of  $\mathbf{G}$ , i.e. a subgroup of  $\mathbf{G}$  which is isomorphic over  $\mathbb{Q}$  (resp.  $\mathbb{R}$ ) to the direct product of  $q$  (resp.  $r \geq q$ ) copies of  $\mathbb{C}^*$ . All such tori are conjugate under  $\mathbf{G}(\mathbb{Q}) = \mathbf{G} \cap \mathbf{GL}(N, \mathbb{Q})$  (resp.  $\mathbf{G}(\mathbb{R})$ ) and their common dimension  $q$  (resp.  $r$ ) is called the  $\mathbb{Q}$ -rank (resp.  $\mathbb{R}$ -rank) of  $\mathbf{G}$ . The identity component of  $\mathbf{S}(\mathbb{R})$  (resp.  $\mathbf{T}(\mathbb{R})$ ) will be denoted by  $A$  (resp.  $A_0$ ), the corresponding Lie algebras by  $\mathfrak{a}$  (resp.  $\mathfrak{a}_0$ ). The  $\mathbb{R}$ -rank of  $\mathbf{G}$  coincides with the rank of the symmetric space  $X$ , i.e. the maximal dimension of totally geodesic flat subspaces. The choice of a maximal compact subgroup  $K$  of  $G$

is equivalent to the choice of a base point  $x_0$  of  $X$ . We can choose  $K$  with Lie algebra  $\mathfrak{k}$  so that under the corresponding Cartan decomposition  $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$  of the Lie algebra  $\mathfrak{g}$  of  $G$  we have  $\mathfrak{a} \subseteq \mathfrak{a}_0 \subset \mathfrak{p} \cong T_{x_0}X$ . Here  $\mathfrak{a}_0$  is maximal abelian in  $\mathfrak{p}$ , i.e. the tangent space at  $x_0$  of the (maximal  $\mathbb{R}$ -) flat  $A_0 \cdot x_0$  in  $X$ . The pair of Lie algebras  $(\mathfrak{g}, \mathfrak{a}_0)$  gives rise to the root system  ${}_{\mathbb{R}}\Phi$  of the symmetric space. Similarly there is a system of  $\mathbb{Q}$ -roots  ${}_{\mathbb{Q}}\Phi$  associated to the pair  $(\mathfrak{g}, \mathfrak{a})$  (see [B3] §21). It is always possible to choose orderings of  ${}_{\mathbb{Q}}\Phi$  and  ${}_{\mathbb{R}}\Phi$  such that the restrictions of simple  $\mathbb{R}$ -roots of  ${}_{\mathbb{R}}\Phi$  to  $\mathfrak{a}$  are either *simple*  $\mathbb{Q}$ -roots of  ${}_{\mathbb{Q}}\Phi$ , i.e. the elements of a basis  $\Delta = {}_{\mathbb{Q}}\Delta$  of  ${}_{\mathbb{Q}}\Phi$ , or zero (see [BT] 6.8). The basis  ${}_{\mathbb{R}}\Delta$  defines a closed  $\mathbb{R}$ -Weyl chamber  $\overline{\mathfrak{a}_0^+}$  in  $\mathfrak{a}_0$  and  $\Delta$  then determines a closed  $\mathbb{Q}$ -Weyl chamber  $\overline{\mathfrak{a}^+} := \{H \in \mathfrak{a} \mid \alpha(H) \geq 0, \text{ for all } \alpha \in \Delta\}$  in  $\mathfrak{a}$ . We set  $\overline{A^+} = \exp \overline{\mathfrak{a}^+}$  (resp.  $\overline{A_0^+} = \exp \overline{\mathfrak{a}_0^+}$ ). A  $\mathbb{Q}$ -Weyl chamber in  $X$  is a translate of the basic chamber  $\overline{A^+} \cdot x_0 \subseteq \overline{A_0^+} \cdot x_0$ . The elements of  $\Delta$  are differentials of characters (defined over  $\mathbb{Q}$ ) of the maximal  $\mathbb{Q}$ -split torus  $\mathbf{S}$ . It is convenient to identify the elements of  $\Delta$  also with such characters. When restricted to  $A$  their values are denoted by  $\alpha(a)$  ( $a \in A, \alpha \in \Delta$ ). Notice that  $\overline{A^+} = \{a \in A \mid \alpha(a) \geq 1 \text{ for all } \alpha \in \Delta\}$ .

A closed subgroup  $\mathbf{P}$  of  $\mathbf{G}$  defined over  $\mathbb{Q}$  is a *parabolic  $\mathbb{Q}$ -subgroup* if  $\mathbf{G}/\mathbf{P}$  is a projective variety (see [B3] §11). A *parabolic  $\mathbb{Q}$ -subgroup*  $P$  of  $G = \mathbf{G}(\mathbb{R})^0$  is by definition the intersection of  $G$  with a parabolic  $\mathbb{Q}$ -subgroup of  $\mathbf{G}$  (see [BS]). The conjugacy classes under  $\mathbf{G}(\mathbb{Q})$  of parabolic  $\mathbb{Q}$ -subgroups are in one-to-one correspondence with the subsets  $\Theta$  of the (chosen) set  $\Delta$  of simple  $\mathbb{Q}$ -roots; they are represented by the *standard parabolic  $\mathbb{Q}$ -subgroups*  $\mathbf{P}_{\Theta}$  of  $\mathbf{G}$  (see [B3] §21.11). The corresponding standard parabolic  $\mathbb{Q}$ -subgroups of  $G$  are denoted by  $P_{\Theta}$ . The minimal parabolic subgroup  $P = P_{\emptyset}$  has a decomposition  $P = UMA$ , where  $U$  is unipotent and  $M$  is reductive;  $A$  centralizes  $M$  and normalizes  $U$  (see [B1]). This yields a (generalized) Iwasawa decomposition for  $G$ , i.e.  $G = P \cdot K = UMAK$ , which implies that  $P$  acts transitively on the symmetric space  $X$ . The intersection of the maximal compact subgroup  $K$  of  $G$  with  $M$  is maximal compact in  $M$  and the quotient  $Z = M/(K \cap M)$  is (in general) the Riemannian product of a symmetric space of noncompact type by a (flat) Euclidean space. Let  $\tau : M \longrightarrow Z$  be the natural projection. Then the “horocyclic coordinate map”

$$\mu : Y = U \times Z \times A \longmapsto X \quad ; \quad (u, \tau(m), a) \longmapsto uma \cdot x_0$$

is an isomorphism of analytic manifolds (see [BS] or [B2]).

A *generalized Siegel set*  $\mathcal{S} = \mathcal{S}_{\omega, \tau}$  in  $X$  (relative to the  $\mathbb{Q}$ -Weyl chamber  $\overline{A^+} \cdot x_0$ ) is a subset of  $X$  of the form  $\mathcal{S}_{\omega, \tau} = \omega A_\tau \cdot x_0$  where  $\omega$  is relatively compact in  $UM$  and, for  $\tau > 0$ ,  $A_\tau = \{a \in A \mid \alpha(a) \geq \tau, \alpha \in \Delta\}$ . If we define  $a_0 \in A$  by  $\alpha(a_0) = \tau$  for all  $\alpha \in \Delta$ , then  $A_\tau = A_1 a_0 = \overline{A^+} a_0$  and  $\mathcal{C} = A_\tau \cdot x_0 \subset \mathcal{S}$  is a (translate of a)  $\mathbb{Q}$ -Weyl chamber in  $X$ . Siegel sets provide the building blocks for (approximate) fundamental domains for arithmetic groups. A subset  $\Omega \subset X$  is called a *fundamental set* for an arithmetic group  $\Gamma$  if the following two conditions hold

- (i)  $X = \Gamma \cdot \Omega$ ;
- (ii) for every  $q \in \mathbf{G}(\mathbb{Q})$  the set  $\{\gamma \in \Gamma \mid q\Omega \cap \gamma\Omega \neq \emptyset\}$  is finite.

The existence of fundamental sets is guaranteed by reduction theory for arithmetic groups (see [B1] §13 and §15).

**PROPOSITION 2.1** (Borel, Harish-Chandra). *Let  $\mathbf{G}$  be a semisimple algebraic group defined over  $\mathbb{Q}$  with associated Riemannian symmetric space  $X = G/K$ . Let  $\mathbf{P}$  be a minimal parabolic  $\mathbb{Q}$ -subgroup of  $\mathbf{G}$  and let  $\Gamma$  be an arithmetic subgroup of  $\mathbf{G}(\mathbb{Q})$ . Then there exists a generalized Siegel set  $\mathcal{S} = \mathcal{S}_{\omega, \tau}$  (with respect to  $\overline{A^+} \cdot x_0$ ) such that, for a (fixed) set  $\{q_i \mid 1 \leq i \leq m\}$  of representatives of the finite set of double cosets  $\Gamma \backslash \mathbf{G}(\mathbb{Q}) / \mathbf{P}(\mathbb{Q})$ , the union  $\Omega = \bigcup_{i=1}^m q_i \cdot \mathcal{S}$  is a fundamental set (of finite volume) for  $\Gamma$  in  $X$ .*

It will be useful in the sequel to dispose of geometric interpretations of the above algebraic concepts and assertions.

First recall that the symmetric space  $X$ , as a Riemannian manifold of nonpositive curvature, has an *ideal boundary at infinity*  $\partial_\infty X$ . The latter is defined as the set of equivalence classes of asymptotic geodesic rays (see [BGS]). In the same way one also defines the ideal boundary at infinity  $\partial_\infty V$  of  $V = \Gamma \backslash X$ . If  $\Gamma$  is an arithmetic lattice in a group  $\mathbf{G}$  of  $\mathbb{Q}$ -rank  $q = 1$ , the boundary  $\partial_\infty V$  of the associated locally symmetric space consists of  $m$  points (corresponding to the cusps), where  $m$  is as in Proposition 2.1. For  $\mathbb{Q}$ -rank  $q \geq 2$  it turns out that  $\partial_\infty V$  is isomorphic to a finite simplicial complex  $\Gamma \backslash |\mathcal{T}|$ , a geometric realization of the Tits building of  $\mathbf{G}$  modulo  $\Gamma$  (see [JM] and [L1]). We recall the construction of the latter.

Let  $\mathcal{P}$  be the set of all parabolic  $\mathbb{Q}$ -subgroups of  $\mathbf{G}$ . The conjugacy classes of elements of  $\mathcal{P}$  are in one-to-one correspondence with the subsets  $\Theta$  of the set  $\Delta$  of simple  $\mathbb{Q}$ -roots. Every conjugacy class has a standard representative denoted by  $\mathbf{P}_\Theta$ . One can show that the sets of double cosets  $\Gamma \backslash \mathbf{G}(\mathbb{Q}) / \mathbf{P}_\Theta(\mathbb{Q})$  are *finite* for all  $\Theta$  (see [B1], §15.6). Let  $\Delta = [e_1, \dots, e_q] \subset \mathbb{R}^q$  denote a

standard geometric  $q-1$  simplex ( $q = \mathbb{Q}$ -rank of  $\mathbf{G}$ ). If  $\Delta = \{\alpha_1, \dots, \alpha_q\}$  and  $\Delta - \Theta = \{\alpha_{i_1}, \dots, \alpha_{i_s}\}$  with  $1 \leq i_1 < \dots < i_s \leq q$ , we define the boundary simplex  $\Delta(\Theta)$  of  $\Delta$  as  $\Delta(\Theta) := [e_{i_1}, \dots, e_{i_s}]$ . Let  $\mathbf{P}$  be a *minimal* parabolic  $\mathbb{Q}$ -subgroup of  $\mathbf{G}$  and let the set  $\Gamma \backslash \mathbf{G}(\mathbb{Q})/\mathbf{P}(\mathbb{Q})$  be represented by  $\{q_1, \dots, q_m\}$  (see Proposition 2.1). We take  $m$  copies  $\Delta^j = [e_1^j, \dots, e_q^j]$  of  $\Delta$  with faces  $\Delta^j(\Theta)$  corresponding to  $\Theta$ . The corresponding homeomorphisms  $\Delta \simeq \Delta^j$  are denoted by  $\varphi_j$ . The simplicial complex  $\Gamma \backslash |\mathcal{T}|$ , which provides a geometric realization of the quotient of the Tits building of  $\mathbf{G}$  modulo  $\Gamma$ , is constructed from the simplices  $\Delta^1, \dots, \Delta^m$  through the following *incidence relations*:

Two simplices  $\Delta^j$  and  $\Delta^l$  are pasted together along the faces  $\Delta^j(\Theta)$  and  $\Delta^l(\Theta)$  by the homeomorphism  $\varphi_j \circ \varphi_l^{-1} |_{\Delta^l(\Theta)}$  if and only if

$$\Gamma q_j \mathbf{P}_\Theta(\mathbb{Q}) = \Gamma q_l \mathbf{P}_\Theta(\mathbb{Q}).$$

We remark that the points of  $\Gamma \backslash |\mathcal{T}|$  are in one-to-one correspondence with equivalence classes of geodesic rays in the locally symmetric space  $V = \Gamma \backslash X$  (see [Hat], [L1] and [JM]).

## 2.2. AN EXHAUSTION BY POLYHEDRA

We index the “edges” of the Weyl chamber  $\overline{\mathfrak{a}^+}$  (or equivalently of  $\overline{A^+} \cdot x_0$ ) by *simple*  $\mathbb{Q}$ -roots. More precisely, the edges of  $\overline{A^+} \cdot x_0$  are given by geodesic rays  $c_\alpha(t) = \exp(tH_\alpha) \cdot x_0$  where  $H_\alpha \in \overline{\mathfrak{a}^+}$ ,  $\|H_\alpha\| = 1$  and  $\beta(H_\alpha) = 0$  for  $\beta \neq \alpha$  ( $\alpha, \beta \in \Delta$ ). We further write  $c_{k\alpha}$  for the edges  $q_k a_0 c_\alpha$  of the chambers  $q_k \mathcal{C}$  in the fundamental set  $\Omega$  (see Section 2.1 for the notation). If a geodesic ray  $c$  represents a point  $z \in \partial_\infty X$  we write  $z = c(\infty)$ . The group  $G$  act naturally on  $\partial_\infty X$  through  $g \cdot c(\infty) = (g \cdot c)(\infty)$ . For every  $\alpha \in \Delta$  the isotropy group of  $c_\alpha(\infty)$  under that action coincides with the (maximal) parabolic subgroups  $P_{\Delta - \{\alpha\}}$  introduced above (see [L2] Lemma 1.2).

To a geodesic ray  $c: [0, \infty) \rightarrow X$  (parametrized by arc-length) which represents a point  $z$  in the ideal boundary  $\partial_\infty X$  of  $X$  is associated a *Busemann function on  $X$  at  $z$*  given by

$$h_z: X \rightarrow \mathbb{R} \quad ; \quad h_z(x) = \lim_{t \rightarrow \infty} [d(x, c(t)) - t].$$

The level sets of a Busemann function are *horospheres*, which foliate the symmetric space. We denote the Busemann functions which correspond to the rays  $c_{k\alpha}$  by  $h_{k\alpha}$ . Note that  $h_{k\alpha}(c_{k\alpha}(t))$  tends to  $-\infty$  if the arc-length  $t$  of the geodesic  $c_{k\alpha}$  tends to  $+\infty$ .

In contrast to an exact fundamental domain there are not only points on the boundary of a fundamental set  $\Omega$  but possibly also interior points which are

identified under the action of  $\Gamma$ . However, there is only a finite set of isometries  $\gamma \in \Gamma$  with  $\gamma\Omega \cap \Omega \neq \emptyset$ . Furthermore it suffices to look at the (finite) set  $\mathcal{D}$  of those  $\gamma$  for which this intersection is not relatively compact in  $X$  (all other intersections are contained in some compact subset of  $\Omega$ ). It turns out that every  $\gamma \in \mathcal{D}$  has the crucial property that there are indices  $i, j$  such that  $q_j^{-1}\gamma q_i$  is parabolic i.e. fixes at least one point in the ideal boundary  $\partial_\infty X$  (see [L2] Proposition 2.2). Then for every  $\gamma \in \mathcal{D}$  there are indices  $i, j, \alpha$  such the family of horospheres of the form  $h_{i\alpha}^{-1}(s), s \in \mathbb{R}$ , is mapped isometrically to the family  $h_{j\alpha}^{-1}(s), s \in \mathbb{R}$  (see [L2] Lemma 3.2). These identifications correspond to the incidence relations described above in the construction of the simplicial complex  $\Gamma \backslash |\mathcal{T}|$ . (To see this one has to use the fact that the Siegel set at infinity  $\partial_\infty(q_j\mathcal{S})$  is canonically isomorphic to  $\Delta^j = [e_1^j, \dots, e_q^j]$ .) The main technical step is then to renormalize the Busemann functions as  $\tilde{h}_{i\alpha} = h_{i\alpha} - s_{ij}$  (for certain constants  $s_{ij}$ ) in such a way that each  $\gamma \in \mathcal{D}$  maps a horosphere of some given level, say  $\{\tilde{h}_{i\alpha} = s\}$ , to another one,  $\{\tilde{h}_{j\alpha} = s\}$ , of the *same* level  $s$  (see [L2] Lemma 3.4). This fact finally allows us to truncate the constituents  $q_i\mathcal{S}$  of the fundamental set  $\Omega$  by removing the open horoballs  $\mathcal{B}_{i\alpha}(s) := \{\tilde{h}_{i\alpha} < -\tau_\alpha s\}$  (for certain constants  $\tau_\alpha$  and for  $s > 0$  sufficiently large). The above construction guarantees that the truncated fundamental set  $\Omega(s) := \bigcup_{i=1}^m q_i\mathcal{S}(s)$  of  $\Omega$  is relatively compact in  $X$  and invariant under the (restricted) action of  $\Gamma$ . Moreover for  $s$  sufficiently large the  $\Gamma$ -invariant “core”  $X(s) := \Gamma \cdot \Omega(s)$  can be written as the complement in  $X$  of a union of (countably many) open horoballs:  $X(s) = X - \Gamma \cdot \bigcup_{i=1}^m \bigcup_{\alpha \in \Delta} \mathcal{B}_{i\alpha}(s)$  (see [L3] Theorem 3.6). These horoballs are disjoint if and only if  $\Gamma$  is an arithmetic subgroup of a  $\mathbb{Q}$ -rank 1 group. The projection  $\pi : X \rightarrow V$  maps  $X(s)$  to a compact submanifold with corners  $V(s)$  of  $V$  whose fundamental group is isomorphic to  $\Gamma$ . The “centers” of the projected horoballs in  $\partial_\infty V$  are in bijection with the vertices of  $\Gamma \backslash |\mathcal{T}|$ . The exhaustion function  $h$  is eventually defined in such a way that its level sets coincide with the boundaries  $\partial V(s)$ . We summarize the result in the following proposition (see [L2] Theorem 4.2).

**PROPOSITION 2.2.** *Let  $X$  be a Riemannian symmetric space of noncompact type and  $\mathbb{R}$ -rank  $\geq 2$  and let  $\Gamma$  be an irreducible, torsion-free, non-uniform lattice in the group of isometries of  $X$ . On the locally symmetric space  $V = \Gamma \backslash X$  there exists a piecewise real analytic exhaustion function  $h : V \rightarrow [0, \infty)$  such that, for each  $s \geq 0$ , the sublevel set  $V(s) := \{h \leq s\}$  is a Riemannian polyhedron in  $V$ . Moreover the level sets  $\{h = s\} = \partial V(s)$  consist of projections of pieces of horospheres in  $X$ .*

Each polyhedron  $V(s)$  is homotopically equivalent to  $V$ . More precisely we have

**PROPOSITION 2.3.** *For every sufficiently large  $s$  the locally symmetric space  $V$  is homeomorphic to the interior of the polyhedron  $V(s)$  in  $V$ , and  $V(s)$  is a strong deformation retract of  $V$ .*

For the proof see [L3], Theorems 5.2 and 5.5.

### 3. ESTIMATES FOR THE BOUNDARY SUBPOLYHEDRA

We wish to apply Proposition 1.1 to the polyhedra  $V(s)$  in the above exhaustion and then take the limit for  $s \rightarrow \infty$ . To that end we need estimates for the second fundamental forms and the volumes of the (lower dimensional) boundary polyhedra.

For each Siegel set  $\mathcal{S}_i := q_i \mathcal{S}$  which is part of the fundamental set  $\Omega$  we have its truncated part

$$\mathcal{S}_i(s) := \mathcal{S}_i - \bigcup_{\alpha \in \Delta} (\mathcal{B}_{i\alpha}(s) \cap \mathcal{S}_i).$$

The top dimensional boundary faces of  $\mathcal{S}_i(s)$  in  $\mathcal{S}_i$  (resp. of  $\Omega(s)$  in  $\Omega$ ) are subsets of horospheres:

$$\mathcal{H}_{i\alpha}(s) := \{\tau_\alpha^{-1} \tilde{h}_{i\alpha} = -s\} \cap \mathcal{S}_i(s), \quad \alpha \in \Delta.$$

The “horospherical” pieces  $\mathcal{H}_{i\alpha}(s)$  together with their  $\Gamma$ -translates form the boundary of the manifold with corners  $X(s)$  in  $X$ . For any nonempty subset  $\Theta$  of  $\Delta$  we set

$$\mathcal{H}_{i\Theta}(s) := \bigcap_{\alpha \in \Theta} \mathcal{H}_{i\alpha}(s) \subset \mathcal{S}_i(s).$$

The various boundary subpolyhedra of  $V(s)$  are then unions of projections of the pieces  $\mathcal{H}_{i\Theta}(s)$  under the canonical projection  $\pi : X \rightarrow V$ . More precisely, as explained in Section 2, for any subset  $\Theta \subset \Delta$ , we have the equivalence relation on the set  $I = \{1, \dots, m\}$

$$j \sim_\Theta l \quad \text{if and only if} \quad \Gamma q_j P_\Theta = \Gamma q_l P_\Theta$$

(the  $q_i$  are as in Proposition 2.1). This relation  $\sim_\Theta$  induces a partition,  $I(\Theta)$ , of the set  $I$  whose components will be denoted by  $E$ . Let  $n = \dim X = \dim V$ , let  $k$  be the cardinality of  $\Theta$  and let  $E \in I(\Theta)$ . Then  $V_E^{n-k}(s) := \pi(\bigcup_{i \in E} \mathcal{H}_{i\Theta}(s))$  is a  $(n - k)$ -dimensional boundary polyhedron of  $V(s)$ ; and moreover, any boundary polyhedron arises in this way (see [L3] §4).