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# ON THE GAUSS-BONNET FORMULA FOR LOCALLY SYMMETRIC SPACES OF NONCOMPACT TYPE

by Enrico LEUZINGER

ABSTRACT. Let $X$ be a Riemannian symmetric space of noncompact type and rank $\geq 2$ and let $\Gamma$ be a non-uniform, irreducible lattice in the group of isometries of $X$. A Gauss-Bonnet formula for the locally symmetric quotient $V = \Gamma\backslash X$ was first proved by G. Harder. We present a new simple proof which is based on an exhaustion of $V$ by Riemannian polyhedra with uniformly bounded second fundamental forms.

## INTRODUCTION

The generalized Gauss-Bonnet theorem of C. B. Allendoerfer, A. Weil and S. S. Chern asserts that the Euler characteristic of a *closed* Riemannian manifold $(M, g)$ is given by

$$\chi(M) = \int_M \omega_g$$

where the Gauss-Bonnet-Chern form $\omega_g = \Psi_g \, dv_g$ is (locally) computable from the metric $g$ (see [AW], [C]).

In several articles J. Cheeger and M. Gromov investigated the Gauss-Bonnet theorem for *open* complete Riemannian manifolds with bounded sectional curvature and finite volume. They in particular showed that such manifolds $M^n$ admit an exhaustion by compact manifolds with smooth boundary, $M_i^n$, such that $\mathrm{Vol}(\partial M_i^n) \to 0$ $(i \to \infty)$ and for which the second fundamental forms $\mathrm{II}(\partial M_i^n)$ are uniformly bounded (see [CG1], [CG2], [CG3] and also [G] 4.5.C). By a formula of Chern one has $\chi(M_i^n) = \int_{M_i^n} \omega_g + \int_{\partial M_i^n} \eta_i$ where $\eta_i$ is a certain form on the boundary $\partial M_i^n$ (see [C]). The above two properties imply that $\lim_{i \to \infty} \int_{\partial M_i^n} \eta_i = 0$ and hence $\chi(M_i^n) = \int_{M^n} \omega_g$ for sufficiently large $i$. As a consequence the Gauss-Bonnet theorem holds whenever $\chi(M_i^n) = \chi(M^n)$ for all sufficiently large $i$.

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We now consider a Riemannian symmetric space $X$ of noncompact type and rank $\geq 2$ and a non-uniform, torsion-free lattice $\Gamma$ in the group of isometries of $X$. The quotient $V = \Gamma\backslash X$ is a locally symmetric space with bounded non-positive sectional curvature and finite volume. Locally symmetric spaces thus provide important examples for the above class considered by Cheeger and Gromov. If $\Gamma$ is irreducible a remarkable theorem of G. A. Margulis asserts that $\Gamma$ is *arithmetic* (see [Z], Ch. 6). For quotients of such lattices the Gauss-Bonnet formula was first proved by G. Harder (see [H]). Following M. S. Raghunathan [R1] he explicitly constructed a smooth exhaustion function $h$ on $V$ which has no critical points outside a compact set. A certain defect of the function $h$, however, is the quite complicated geometry of its sublevel sets (their second fundamental forms, for instance, are not uniformly bounded). As a consequence the proof given in [H] involves rather long and technical estimates.

The purpose of the present note is two-fold. On the one hand to give a new, more geometric proof of the Gauss-Bonnet theorem for locally symmetric spaces, which avoids the technically complicated estimates of [H]. And, on the other hand, to provide an explicit (and independent) illustration of general results in [CG3].

Our approach is based on an exhaustion $V = \bigcup_{s\geq 0} V(s)$ of locally symmetric spaces *not* by *smooth* submanifolds but by *polyhedra*, i.e. compact submanifolds with corners (see [L2]). The corners which appear here are naturally related to the geometry of $V$ at infinity (and therefore should not be smoothed). Moreover, for each $s \geq 0$ the polyhedron $V(s)$ is a strong deformation retract of $V$ (see [L3]). The essential new feature of this exhaustion is that the boundaries of $\partial V(s)$ consist of subpolyhedra of $V(s)$ which are projections of pieces of horospheres in $X$. As a consequence their second fundamental forms are uniformly bounded. This property together with the generalized Gauss-Bonnet formula for Riemannian polyhedra of Allendoerfer-Weil and Chern leads to a considerably simplified new proof of the Gauss-Bonnet theorem for locally symmetric spaces (see Theorem 4.1).

NOTATION. Explicit constants are irrelevant for our purpose. If $f$ and $g$ are positive real valued functions on a set $S$ we thus simply write $f \prec g$ if there is a constant $c > 0$ such that $f(s) \leq cg(s)$ for all $s \in S$.