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## ON THE GAUSS-BONNET FORMULA FOR LOCALLY SYMMETRIC SPACES OF NONCOMPACT TYPE

by Enrico LEUZINGER

ABSTRACT. Let  $X$  be a Riemannian symmetric space of noncompact type and rank  $\geq 2$  and let  $\Gamma$  be a non-uniform, irreducible lattice in the group of isometries of  $X$ . A Gauss-Bonnet formula for the locally symmetric quotient  $V = \Gamma \backslash X$  was first proved by G. Harder. We present a new simple proof which is based on an exhaustion of  $V$  by Riemannian polyhedra with uniformly bounded second fundamental forms.

### INTRODUCTION

The generalized Gauss-Bonnet theorem of C.B. Allendoerfer, A. Weil and S.S. Chern asserts that the Euler characteristic of a *closed* Riemannian manifold  $(M, g)$  is given by

$$\chi(M) = \int_M \omega_g$$

where the Gauss-Bonnet-Chern form  $\omega_g = \Psi_g dv_g$  is (locally) computable from the metric  $g$  (see [AW], [C]).

In several articles J. Cheeger and M. Gromov investigated the Gauss-Bonnet theorem for *open* complete Riemannian manifolds with bounded sectional curvature and finite volume. They in particular showed that such manifolds  $M^n$  admit an exhaustion by compact manifolds with smooth boundary,  $M_i^n$ , such that  $\text{Vol}(\partial M_i^n) \rightarrow 0$  ( $i \rightarrow \infty$ ) and for which the second fundamental forms  $\text{II}(\partial M_i^n)$  are uniformly bounded (see [CG1], [CG2], [CG3] and also [G] 4.5.C). By a formula of Chern one has  $\chi(M_i^n) = \int_{M_i^n} \omega_g + \int_{\partial M_i^n} \eta_i$  where  $\eta_i$  is a certain form on the boundary  $\partial M_i^n$  (see [C]). The above two properties imply that  $\lim_{i \rightarrow \infty} \int_{\partial M_i^n} \eta_i = 0$  and hence  $\chi(M_i^n) = \int_{M^n} \omega_g$  for sufficiently large  $i$ . As a consequence the Gauss-Bonnet theorem holds whenever  $\chi(M_i^n) = \chi(M^n)$  for all sufficiently large  $i$ .

We now consider a Riemannian symmetric space  $X$  of noncompact type and rank  $\geq 2$  and a non-uniform, torsion-free lattice  $\Gamma$  in the group of isometries of  $X$ . The quotient  $V = \Gamma \backslash X$  is a locally symmetric space with bounded non-positive sectional curvature and finite volume. Locally symmetric spaces thus provide important examples for the above class considered by Cheeger and Gromov. If  $\Gamma$  is irreducible a remarkable theorem of G. A. Margulis asserts that  $\Gamma$  is *arithmetic* (see [Z], Ch. 6). For quotients of such lattices the Gauss-Bonnet formula was first proved by G. Harder (see [H]). Following M. S. Raghunathan [R1] he explicitly constructed a smooth exhaustion function  $h$  on  $V$  which has no critical points outside a compact set. A certain defect of the function  $h$ , however, is the quite complicated geometry of its sublevel sets (their second fundamental forms, for instance, are not uniformly bounded). As a consequence the proof given in [H] involves rather long and technical estimates.

The purpose of the present note is two-fold. On the one hand to give a new, more geometric proof of the Gauss-Bonnet theorem for locally symmetric spaces, which avoids the technically complicated estimates of [H]. And, on the other hand, to provide an explicit (and independent) illustration of general results in [CG3].

Our approach is based on an exhaustion  $V = \bigcup_{s \geq 0} V(s)$  of locally symmetric spaces *not* by *smooth* submanifolds but by *polyhedra*, i.e. compact submanifolds with corners (see [L2]). The corners which appear here are naturally related to the geometry of  $V$  at infinity (and therefore should not be smoothed). Moreover, for each  $s \geq 0$  the polyhedron  $V(s)$  is a strong deformation retract of  $V$  (see [L3]). The essential new feature of this exhaustion is that the boundaries of  $\partial V(s)$  consist of subpolyhedra of  $V(s)$  which are projections of pieces of horospheres in  $X$ . As a consequence their second fundamental forms are uniformly bounded. This property together with the generalized Gauss-Bonnet formula for Riemannian polyhedra of Allendoerfer-Weil and Chern leads to a considerably simplified new proof of the Gauss-Bonnet theorem for locally symmetric spaces (see Theorem 4.1).

NOTATION. Explicit constants are irrelevant for our purpose. If  $f$  and  $g$  are positive real valued functions on a set  $S$  we thus simply write  $f \prec g$  if there is a constant  $c > 0$  such that  $f(s) \leq cg(s)$  for all  $s \in S$ .

## 1. THE FORMULA OF ALLENDOERFER AND WEIL

A  $C^\infty$  (resp.  $C^\omega$ ) manifold with corners is a topological Hausdorff space locally modeled upon a product of lines and half-lines and such that coordinate changes are of class  $C^\infty$  (resp.  $C^\omega$ ). For precise definitions and basic information about this concept we refer to [DH]. A *Riemannian polyhedron* is a compact manifold with corners equipped with a Riemannian metric.

Let  $\mathcal{P}^n$  be an  $n$ -dimensional Riemannian polyhedron with boundary consisting of a finite family of lower dimensional subpolyhedra

$$\mathcal{P}_E^{n-k} \quad (0 \leq k \leq n-1)$$

and with Riemannian metric induced from  $\mathcal{P}^n$ . The *outer angle*  $O(p)$  at a point  $p$  of  $\mathcal{P}_E^{n-k}$  is defined as the set of all unit tangent vectors  $v \in T_p \mathcal{P}^n$  such that  $\langle v, w \rangle_p \leq 0$  for all  $w$  in the tangent cone of  $\mathcal{P}^n$  at  $p$ . Note that  $O(p)$  is a spherical cell bounded by “great spheres” in the  $(k-1)$ -dimensional unit sphere of the normal space of  $\mathcal{P}_E^{n-k} \subset \mathcal{P}^n$  at  $p$ . In [AW] Allendoerfer and Weil define a certain real valued function  $\Psi_{E,k}$  on the outer angles of  $\mathcal{P}_E^{n-k}$ . The explicit form of this function will not be needed in this paper. We shall only use the fact that  $\Psi_{E,k}$  is locally computable from the components of the metric and the curvature tensor of  $\mathcal{P}^n$  and from the components of the second fundamental forms  $\Pi_Z(p), Z \in O(p)$ , of  $\mathcal{P}_E^{n-k}$  in  $\mathcal{P}^n$ . Let  $\Psi dv$  denote the Gauss-Bonnet-Chern form on  $\mathcal{P}^n$  and  $dv_E$  (resp.  $d\omega_{k-1}$ ) the volume element of  $\mathcal{P}_E^k$  (resp. of the standard unit sphere  $S^{k-1}$ ). The *inner Euler characteristic*  $\chi'$  of  $\mathcal{P}^n$  is by definition the Euler characteristic of the open complex consisting of all inner cells in an arbitrary simplicial subdivision of  $\mathcal{P}^n$ .

We can now state the generalized Gauss-Bonnet formula of Allendoerfer-Weil for Riemannian polyhedra (see [AW]).

**PROPOSITION 1.1.** *Let  $\mathcal{P}^n$  be a Riemannian polyhedron with boundary consisting of a finite family of subpolyhedra  $\mathcal{P}_E^{n-k}$ . Then the inner Euler characteristic of  $\mathcal{P}^n$  is given by*

$$(-1)^n \chi'(\mathcal{P}^n) = \int_{\mathcal{P}^n} \Psi dv + \sum_{k=1}^n \sum_E \int_{\mathcal{P}_E^{n-k}} \left( \int_{O(p)} \Psi_{E,k} d\omega_{k-1} \right) dv_E(p).$$



## 2. AN EXHAUSTION OF LOCALLY SYMMETRIC SPACES

Let  $X$  be a Riemannian symmetric space of noncompact type and rank  $\geq 2$  and let  $\Gamma$  be a non-uniform, torsion-free lattice in the group of isometries of  $X$ . In this section we briefly describe the basic features of an exhaustion of the locally symmetric space  $V = \Gamma \backslash X$  by Riemannian polyhedra, which was previously constructed in [L2].

The idea is to work with a *fundamental set*  $\Omega \subset X$  for the discrete (arithmetic) group  $\Gamma$ . Such “coarse” fundamental domains are provided by *reduction theory*; they are finite unions of translates of so-called Siegel sets. We begin with reviewing some facts about linear algebraic groups and set up the notation. Roughly speaking, the lattice  $\Gamma$  determines a “ $\mathbb{Q}$ -structure” on the real Lie group of isometries of  $X$  such that  $\Gamma$  is given by integer matrices. The symmetric space  $X$  in turn inherits canonical parametrizations adopted to this structure (generalized horocyclic coordinates). Siegel sets are then defined with respect to such parametrizations.

## 2.1. REDUCTION THEORY AND GEOMETRY AT INFINITY

We denote by  $G$  the identity component of the group of isometries of  $X$ ; it is a connected, semisimple Lie group with trivial center. We shall always assume in the following that the non-uniform lattice  $\Gamma$  is *irreducible* (see [R2] 5.20). Then, by the arithmeticity theorem of Margulis, there is a connected semisimple linear algebraic group  $\mathbf{G}$  defined over  $\mathbb{Q}$ ,  $\mathbb{Q}$ -embedded in a general linear group  $\mathbf{GL}(N, \mathbb{C})$ , and a Lie group isomorphism  $p : G \longrightarrow \mathbf{G}(\mathbb{R})^0$  such that  $p(\Gamma)$  is *arithmetic*, i.e.  $p(\Gamma) \subset \mathbf{G}(\mathbb{Q}) \subset \mathbf{GL}(N, \mathbb{C})$  is commensurable with the group  $\mathbf{G}(\mathbb{Z}) = \mathbf{G} \cap \mathbf{GL}(N, \mathbb{Z})$  (see [Z] 3.1.6 and 6.1.10). The symmetric space  $X$  can be recovered as the manifold of maximal compact subgroups of the identity component of the group  $\mathbf{G}(\mathbb{R}) = \mathbf{G} \cap \mathbf{GL}(N, \mathbb{R})$  of  $\mathbb{R}$ -rational points of  $\mathbf{G}$ . For simplicity we will always identify  $G$  with  $\mathbf{G}(\mathbb{R})^0$  and  $\Gamma$  with  $p(\Gamma)$ .

Let  $\mathbf{S}$  (resp.  $\mathbf{T}$ ) be a maximal  $\mathbb{Q}$ -split (resp.  $\mathbb{R}$ -split) algebraic torus of  $\mathbf{G}$ , i.e. a subgroup of  $\mathbf{G}$  which is isomorphic over  $\mathbb{Q}$  (resp.  $\mathbb{R}$ ) to the direct product of  $q$  (resp.  $r \geq q$ ) copies of  $\mathbb{C}^*$ . All such tori are conjugate under  $\mathbf{G}(\mathbb{Q}) = \mathbf{G} \cap \mathbf{GL}(N, \mathbb{Q})$  (resp.  $\mathbf{G}(\mathbb{R})$ ) and their common dimension  $q$  (resp.  $r$ ) is called the  $\mathbb{Q}$ -rank (resp.  $\mathbb{R}$ -rank) of  $\mathbf{G}$ . The identity component of  $\mathbf{S}(\mathbb{R})$  (resp.  $\mathbf{T}(\mathbb{R})$ ) will be denoted by  $A$  (resp.  $A_0$ ), the corresponding Lie algebras by  $\mathfrak{a}$  (resp.  $\mathfrak{a}_0$ ). The  $\mathbb{R}$ -rank of  $\mathbf{G}$  coincides with the rank of the symmetric space  $X$ , i.e. the maximal dimension of totally geodesic flat subspaces. The choice of a maximal compact subgroup  $K$  of  $G$

is equivalent to the choice of a base point  $x_0$  of  $X$ . We can choose  $K$  with Lie algebra  $\mathfrak{k}$  so that under the corresponding Cartan decomposition  $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$  of the Lie algebra  $\mathfrak{g}$  of  $G$  we have  $\mathfrak{a} \subseteq \mathfrak{a}_0 \subset \mathfrak{p} \cong T_{x_0}X$ . Here  $\mathfrak{a}_0$  is maximal abelian in  $\mathfrak{p}$ , i.e. the tangent space at  $x_0$  of the (maximal  $\mathbb{R}$ -) flat  $A_0 \cdot x_0$  in  $X$ . The pair of Lie algebras  $(\mathfrak{g}, \mathfrak{a}_0)$  gives rise to the root system  ${}_{\mathbb{R}}\Phi$  of the symmetric space. Similarly there is a system of  $\mathbb{Q}$ -roots  ${}_{\mathbb{Q}}\Phi$  associated to the pair  $(\mathfrak{g}, \mathfrak{a})$  (see [B3] §21). It is always possible to choose orderings of  ${}_{\mathbb{Q}}\Phi$  and  ${}_{\mathbb{R}}\Phi$  such that the restrictions of simple  $\mathbb{R}$ -roots of  ${}_{\mathbb{R}}\Phi$  to  $\mathfrak{a}$  are either *simple*  $\mathbb{Q}$ -roots of  ${}_{\mathbb{Q}}\Phi$ , i.e. the elements of a basis  $\Delta = {}_{\mathbb{Q}}\Delta$  of  ${}_{\mathbb{Q}}\Phi$ , or zero (see [BT] 6.8). The basis  ${}_{\mathbb{R}}\Delta$  defines a closed  $\mathbb{R}$ -Weyl chamber  $\overline{\mathfrak{a}_0^+}$  in  $\mathfrak{a}_0$  and  $\Delta$  then determines a closed  $\mathbb{Q}$ -Weyl chamber  $\overline{\mathfrak{a}^+} := \{H \in \mathfrak{a} \mid \alpha(H) \geq 0, \text{ for all } \alpha \in \Delta\}$  in  $\mathfrak{a}$ . We set  $\overline{A^+} = \exp \overline{\mathfrak{a}^+}$  (resp.  $\overline{A_0^+} = \exp \overline{\mathfrak{a}_0^+}$ ). A  $\mathbb{Q}$ -Weyl chamber in  $X$  is a translate of the basic chamber  $\overline{A^+} \cdot x_0 \subseteq \overline{A_0^+} \cdot x_0$ . The elements of  $\Delta$  are differentials of characters (defined over  $\mathbb{Q}$ ) of the maximal  $\mathbb{Q}$ -split torus  $S$ . It is convenient to identify the elements of  $\Delta$  also with such characters. When restricted to  $A$  their values are denoted by  $\alpha(a)$  ( $a \in A, \alpha \in \Delta$ ). Notice that  $\overline{A^+} = \{a \in A \mid \alpha(a) \geq 1 \text{ for all } \alpha \in \Delta\}$ .

A closed subgroup  $\mathbf{P}$  of  $\mathbf{G}$  defined over  $\mathbb{Q}$  is a *parabolic  $\mathbb{Q}$ -subgroup* if  $\mathbf{G}/\mathbf{P}$  is a projective variety (see [B3] §11). A *parabolic  $\mathbb{Q}$ -subgroup*  $P$  of  $G = \mathbf{G}(\mathbb{R})^0$  is by definition the intersection of  $G$  with a parabolic  $\mathbb{Q}$ -subgroup of  $\mathbf{G}$  (see [BS]). The conjugacy classes under  $\mathbf{G}(\mathbb{Q})$  of parabolic  $\mathbb{Q}$ -subgroups are in one-to-one correspondence with the subsets  $\Theta$  of the (chosen) set  $\Delta$  of simple  $\mathbb{Q}$ -roots; they are represented by the *standard parabolic  $\mathbb{Q}$ -subgroups*  $\mathbf{P}_{\Theta}$  of  $\mathbf{G}$  (see [B3] §21.11). The corresponding standard parabolic  $\mathbb{Q}$ -subgroups of  $G$  are denoted by  $P_{\Theta}$ . The minimal parabolic subgroup  $P = P_{\emptyset}$  has a decomposition  $P = UMA$ , where  $U$  is unipotent and  $M$  is reductive;  $A$  centralizes  $M$  and normalizes  $U$  (see [B1]). This yields a (generalized) Iwasawa decomposition for  $G$ , i.e.  $G = P \cdot K = UMAK$ , which implies that  $P$  acts transitively on the symmetric space  $X$ . The intersection of the maximal compact subgroup  $K$  of  $G$  with  $M$  is maximal compact in  $M$  and the quotient  $Z = M/(K \cap M)$  is (in general) the Riemannian product of a symmetric space of noncompact type by a (flat) Euclidean space. Let  $\tau : M \longrightarrow Z$  be the natural projection. Then the “horocyclic coordinate map”

$$\mu : Y = U \times Z \times A \longmapsto X \quad ; \quad (u, \tau(m), a) \longmapsto uma \cdot x_0$$

is an isomorphism of analytic manifolds (see [BS] or [B2]).

A *generalized Siegel set*  $\mathcal{S} = \mathcal{S}_{\omega, \tau}$  in  $X$  (relative to the  $\mathbb{Q}$ -Weyl chamber  $\overline{A^+} \cdot x_0$ ) is a subset of  $X$  of the form  $\mathcal{S}_{\omega, \tau} = \omega A_\tau \cdot x_0$  where  $\omega$  is relatively compact in  $UM$  and, for  $\tau > 0$ ,  $A_\tau = \{a \in A \mid \alpha(a) \geq \tau, \alpha \in \Delta\}$ . If we define  $a_0 \in A$  by  $\alpha(a_0) = \tau$  for all  $\alpha \in \Delta$ , then  $A_\tau = A_1 a_0 = \overline{A^+} a_0$  and  $\mathcal{C} = A_\tau \cdot x_0 \subset \mathcal{S}$  is a (translate of a)  $\mathbb{Q}$ -Weyl chamber in  $X$ . Siegel sets provide the building blocks for (approximate) fundamental domains for arithmetic groups. A subset  $\Omega \subset X$  is called a *fundamental set* for an arithmetic group  $\Gamma$  if the following two conditions hold

- (i)  $X = \Gamma \cdot \Omega$ ;
- (ii) for every  $q \in \mathbf{G}(\mathbb{Q})$  the set  $\{\gamma \in \Gamma \mid q\Omega \cap \gamma\Omega \neq \emptyset\}$  is finite.

The existence of fundamental sets is guaranteed by reduction theory for arithmetic groups (see [B1] §13 and §15).

**PROPOSITION 2.1** (Borel, Harish-Chandra). *Let  $\mathbf{G}$  be a semisimple algebraic group defined over  $\mathbb{Q}$  with associated Riemannian symmetric space  $X = G/K$ . Let  $\mathbf{P}$  be a minimal parabolic  $\mathbb{Q}$ -subgroup of  $\mathbf{G}$  and let  $\Gamma$  be an arithmetic subgroup of  $\mathbf{G}(\mathbb{Q})$ . Then there exists a generalized Siegel set  $\mathcal{S} = \mathcal{S}_{\omega, \tau}$  (with respect to  $\overline{A^+} \cdot x_0$ ) such that, for a (fixed) set  $\{q_i \mid 1 \leq i \leq m\}$  of representatives of the finite set of double cosets  $\Gamma \backslash \mathbf{G}(\mathbb{Q}) / \mathbf{P}(\mathbb{Q})$ , the union  $\Omega = \bigcup_{i=1}^m q_i \cdot \mathcal{S}$  is a fundamental set (of finite volume) for  $\Gamma$  in  $X$ .*

It will be useful in the sequel to dispose of geometric interpretations of the above algebraic concepts and assertions.

First recall that the symmetric space  $X$ , as a Riemannian manifold of nonpositive curvature, has an *ideal boundary at infinity*  $\partial_\infty X$ . The latter is defined as the set of equivalence classes of asymptotic geodesic rays (see [BGS]). In the same way one also defines the ideal boundary at infinity  $\partial_\infty V$  of  $V = \Gamma \backslash X$ . If  $\Gamma$  is an arithmetic lattice in a group  $\mathbf{G}$  of  $\mathbb{Q}$ -rank  $q = 1$ , the boundary  $\partial_\infty V$  of the associated locally symmetric space consists of  $m$  points (corresponding to the cusps), where  $m$  is as in Proposition 2.1. For  $\mathbb{Q}$ -rank  $q \geq 2$  it turns out that  $\partial_\infty V$  is isomorphic to a finite simplicial complex  $\Gamma \backslash |\mathcal{T}|$ , a geometric realization of the Tits building of  $\mathbf{G}$  modulo  $\Gamma$  (see [JM] and [L1]). We recall the construction of the latter.

Let  $\mathcal{P}$  be the set of all parabolic  $\mathbb{Q}$ -subgroups of  $\mathbf{G}$ . The conjugacy classes of elements of  $\mathcal{P}$  are in one-to-one correspondence with the subsets  $\Theta$  of the set  $\Delta$  of simple  $\mathbb{Q}$ -roots. Every conjugacy class has a standard representative denoted by  $\mathbf{P}_\Theta$ . One can show that the sets of double cosets  $\Gamma \backslash \mathbf{G}(\mathbb{Q}) / \mathbf{P}_\Theta(\mathbb{Q})$  are *finite* for all  $\Theta$  (see [B1], §15.6). Let  $\Delta = [e_1, \dots, e_q] \subset \mathbb{R}^q$  denote a

standard geometric  $q-1$  simplex ( $q = \mathbb{Q}$ -rank of  $\mathbf{G}$ ). If  $\Delta = \{\alpha_1, \dots, \alpha_q\}$  and  $\Delta - \Theta = \{\alpha_{i_1}, \dots, \alpha_{i_s}\}$  with  $1 \leq i_1 < \dots < i_s \leq q$ , we define the boundary simplex  $\Delta(\Theta)$  of  $\Delta$  as  $\Delta(\Theta) := [e_{i_1}, \dots, e_{i_s}]$ . Let  $\mathbf{P}$  be a *minimal* parabolic  $\mathbb{Q}$ -subgroup of  $\mathbf{G}$  and let the set  $\Gamma \backslash \mathbf{G}(\mathbb{Q})/\mathbf{P}(\mathbb{Q})$  be represented by  $\{q_1, \dots, q_m\}$  (see Proposition 2.1). We take  $m$  copies  $\Delta^j = [e_1^j, \dots, e_q^j]$  of  $\Delta$  with faces  $\Delta^j(\Theta)$  corresponding to  $\Theta$ . The corresponding homeomorphisms  $\Delta \simeq \Delta^j$  are denoted by  $\varphi_j$ . The simplicial complex  $\Gamma \backslash |\mathcal{T}|$ , which provides a geometric realization of the quotient of the Tits building of  $\mathbf{G}$  modulo  $\Gamma$ , is constructed from the simplices  $\Delta^1, \dots, \Delta^m$  through the following *incidence relations*:

Two simplices  $\Delta^j$  and  $\Delta^l$  are pasted together along the faces  $\Delta^j(\Theta)$  and  $\Delta^l(\Theta)$  by the homeomorphism  $\varphi_j \circ \varphi_l^{-1} |_{\Delta^l(\Theta)}$  if and only if

$$\Gamma q_j \mathbf{P}_\Theta(\mathbb{Q}) = \Gamma q_l \mathbf{P}_\Theta(\mathbb{Q}).$$

We remark that the points of  $\Gamma \backslash |\mathcal{T}|$  are in one-to-one correspondence with equivalence classes of geodesic rays in the locally symmetric space  $V = \Gamma \backslash X$  (see [Hat], [L1] and [JM]).

## 2.2. AN EXHAUSTION BY POLYHEDRA

We index the “edges” of the Weyl chamber  $\overline{\mathfrak{a}^+}$  (or equivalently of  $\overline{A^+} \cdot x_0$ ) by *simple*  $\mathbb{Q}$ -roots. More precisely, the edges of  $\overline{A^+} \cdot x_0$  are given by geodesic rays  $c_\alpha(t) = \exp(tH_\alpha) \cdot x_0$  where  $H_\alpha \in \overline{\mathfrak{a}^+}$ ,  $\|H_\alpha\| = 1$  and  $\beta(H_\alpha) = 0$  for  $\beta \neq \alpha$  ( $\alpha, \beta \in \Delta$ ). We further write  $c_{k\alpha}$  for the edges  $q_k a_0 c_\alpha$  of the chambers  $q_k \mathcal{C}$  in the fundamental set  $\Omega$  (see Section 2.1 for the notation). If a geodesic ray  $c$  represents a point  $z \in \partial_\infty X$  we write  $z = c(\infty)$ . The group  $G$  act naturally on  $\partial_\infty X$  through  $g \cdot c(\infty) = (g \cdot c)(\infty)$ . For every  $\alpha \in \Delta$  the isotropy group of  $c_\alpha(\infty)$  under that action coincides with the (maximal) parabolic subgroups  $P_{\Delta - \{\alpha\}}$  introduced above (see [L2] Lemma 1.2).

To a geodesic ray  $c: [0, \infty) \rightarrow X$  (parametrized by arc-length) which represents a point  $z$  in the ideal boundary  $\partial_\infty X$  of  $X$  is associated a *Busemann function on  $X$  at  $z$*  given by

$$h_z: X \rightarrow \mathbb{R} \quad ; \quad h_z(x) = \lim_{t \rightarrow \infty} [d(x, c(t)) - t].$$

The level sets of a Busemann function are *horospheres*, which foliate the symmetric space. We denote the Busemann functions which correspond to the rays  $c_{k\alpha}$  by  $h_{k\alpha}$ . Note that  $h_{k\alpha}(c_{k\alpha}(t))$  tends to  $-\infty$  if the arc-length  $t$  of the geodesic  $c_{k\alpha}$  tends to  $+\infty$ .

In contrast to an exact fundamental domain there are not only points on the boundary of a fundamental set  $\Omega$  but possibly also interior points which are

identified under the action of  $\Gamma$ . However, there is only a finite set of isometries  $\gamma \in \Gamma$  with  $\gamma\Omega \cap \Omega \neq \emptyset$ . Furthermore it suffices to look at the (finite) set  $\mathcal{D}$  of those  $\gamma$  for which this intersection is not relatively compact in  $X$  (all other intersections are contained in some compact subset of  $\Omega$ ). It turns out that every  $\gamma \in \mathcal{D}$  has the crucial property that there are indices  $i, j$  such that  $q_j^{-1}\gamma q_i$  is parabolic i.e. fixes at least one point in the ideal boundary  $\partial_\infty X$  (see [L2] Proposition 2.2). Then for every  $\gamma \in \mathcal{D}$  there are indices  $i, j, \alpha$  such the family of horospheres of the form  $h_{i\alpha}^{-1}(s), s \in \mathbb{R}$ , is mapped isometrically to the family  $h_{j\alpha}^{-1}(s), s \in \mathbb{R}$  (see [L2] Lemma 3.2). These identifications correspond to the incidence relations described above in the construction of the simplicial complex  $\Gamma \backslash |\mathcal{T}|$ . (To see this one has to use the fact that the Siegel set at infinity  $\partial_\infty(q_j\mathcal{S})$  is canonically isomorphic to  $\Delta^j = [e_1^j, \dots, e_q^j]$ .) The main technical step is then to renormalize the Busemann functions as  $\tilde{h}_{i\alpha} = h_{i\alpha} - s_{ij}$  (for certain constants  $s_{ij}$ ) in such a way that each  $\gamma \in \mathcal{D}$  maps a horosphere of some given level, say  $\{\tilde{h}_{i\alpha} = s\}$ , to another one,  $\{\tilde{h}_{j\alpha} = s\}$ , of the *same* level  $s$  (see [L2] Lemma 3.4). This fact finally allows us to truncate the constituents  $q_i\mathcal{S}$  of the fundamental set  $\Omega$  by removing the open horoballs  $\mathcal{B}_{i\alpha}(s) := \{\tilde{h}_{i\alpha} < -\tau_\alpha s\}$  (for certain constants  $\tau_\alpha$  and for  $s > 0$  sufficiently large). The above construction guarantees that the truncated fundamental set  $\Omega(s) := \bigcup_{i=1}^m q_i\mathcal{S}(s)$  of  $\Omega$  is relatively compact in  $X$  and invariant under the (restricted) action of  $\Gamma$ . Moreover for  $s$  sufficiently large the  $\Gamma$ -invariant “core”  $X(s) := \Gamma \cdot \Omega(s)$  can be written as the complement in  $X$  of a union of (countably many) open horoballs:  $X(s) = X - \Gamma \cdot \bigcup_{i=1}^m \bigcup_{\alpha \in \Delta} \mathcal{B}_{i\alpha}(s)$  (see [L3] Theorem 3.6). These horoballs are disjoint if and only if  $\Gamma$  is an arithmetic subgroup of a  $\mathbb{Q}$ -rank 1 group. The projection  $\pi : X \rightarrow V$  maps  $X(s)$  to a compact submanifold with corners  $V(s)$  of  $V$  whose fundamental group is isomorphic to  $\Gamma$ . The “centers” of the projected horoballs in  $\partial_\infty V$  are in bijection with the vertices of  $\Gamma \backslash |\mathcal{T}|$ . The exhaustion function  $h$  is eventually defined in such a way that its level sets coincide with the boundaries  $\partial V(s)$ . We summarize the result in the following proposition (see [L2] Theorem 4.2).

**PROPOSITION 2.2.** *Let  $X$  be a Riemannian symmetric space of noncompact type and  $\mathbb{R}$ -rank  $\geq 2$  and let  $\Gamma$  be an irreducible, torsion-free, non-uniform lattice in the group of isometries of  $X$ . On the locally symmetric space  $V = \Gamma \backslash X$  there exists a piecewise real analytic exhaustion function  $h : V \rightarrow [0, \infty)$  such that, for each  $s \geq 0$ , the sublevel set  $V(s) := \{h \leq s\}$  is a Riemannian polyhedron in  $V$ . Moreover the level sets  $\{h = s\} = \partial V(s)$  consist of projections of pieces of horospheres in  $X$ .*

Each polyhedron  $V(s)$  is homotopically equivalent to  $V$ . More precisely we have

**PROPOSITION 2.3.** *For every sufficiently large  $s$  the locally symmetric space  $V$  is homeomorphic to the interior of the polyhedron  $V(s)$  in  $V$ , and  $V(s)$  is a strong deformation retract of  $V$ .*

For the proof see [L3], Theorems 5.2 and 5.5.

### 3. ESTIMATES FOR THE BOUNDARY SUBPOLYHEDRA

We wish to apply Proposition 1.1 to the polyhedra  $V(s)$  in the above exhaustion and then take the limit for  $s \rightarrow \infty$ . To that end we need estimates for the second fundamental forms and the volumes of the (lower dimensional) boundary polyhedra.

For each Siegel set  $\mathcal{S}_i := q_i \mathcal{S}$  which is part of the fundamental set  $\Omega$  we have its truncated part

$$\mathcal{S}_i(s) := \mathcal{S}_i - \bigcup_{\alpha \in \Delta} (\mathcal{B}_{i\alpha}(s) \cap \mathcal{S}_i).$$

The top dimensional boundary faces of  $\mathcal{S}_i(s)$  in  $\mathcal{S}_i$  (resp. of  $\Omega(s)$  in  $\Omega$ ) are subsets of horospheres:

$$\mathcal{H}_{i\alpha}(s) := \{\tau_\alpha^{-1} \tilde{h}_{i\alpha} = -s\} \cap \mathcal{S}_i(s), \quad \alpha \in \Delta.$$

The “horospherical” pieces  $\mathcal{H}_{i\alpha}(s)$  together with their  $\Gamma$ -translates form the boundary of the manifold with corners  $X(s)$  in  $X$ . For any nonempty subset  $\Theta$  of  $\Delta$  we set

$$\mathcal{H}_{i\Theta}(s) := \bigcap_{\alpha \in \Theta} \mathcal{H}_{i\alpha}(s) \subset \mathcal{S}_i(s).$$

The various boundary subpolyhedra of  $V(s)$  are then unions of projections of the pieces  $\mathcal{H}_{i\Theta}(s)$  under the canonical projection  $\pi : X \rightarrow V$ . More precisely, as explained in Section 2, for any subset  $\Theta \subset \Delta$ , we have the equivalence relation on the set  $I = \{1, \dots, m\}$

$$j \sim_\Theta l \quad \text{if and only if} \quad \Gamma q_j P_\Theta = \Gamma q_l P_\Theta$$

(the  $q_i$  are as in Proposition 2.1). This relation  $\sim_\Theta$  induces a partition,  $I(\Theta)$ , of the set  $I$  whose components will be denoted by  $E$ . Let  $n = \dim X = \dim V$ , let  $k$  be the cardinality of  $\Theta$  and let  $E \in I(\Theta)$ . Then  $V_E^{n-k}(s) := \pi(\bigcup_{i \in E} \mathcal{H}_{i\Theta}(s))$  is a  $(n - k)$ -dimensional boundary polyhedron of  $V(s)$ ; and moreover, any boundary polyhedron arises in this way (see [L3] §4).



REMARK. The minimal possible dimension which occurs is  $n - q$  where  $q$  is the  $\mathbb{Q}$ -rank of  $\mathbf{G}$ . It is also interesting to note (though not needed below) that the outer angles are isomorphic to  $\mathbb{Q}$ -Weyl chambers and their walls at infinity.

We shall use the following well-known fact about Jacobi fields in symmetric spaces (see [K] Theorem 2.2.9). A Jacobi field along a geodesic ray is called *stable* if its length is bounded.

LEMMA 3.1. *Let  $r : [0, \infty) \rightarrow X$  be a unit-speed geodesic ray in the symmetric space  $X$  (of noncompact type). Set  $p = r(0)$ . Then the unique stable Jacobi field  $J_u(s)$  along  $r(s)$  with  $J_u(0) = u \in T_p X$  can be written as*

$$J_u(s) = \sum_j e^{-\lambda_j s} a_j E_j(s)$$

where  $\{E_j(s)\}$  is an orthonormal frame of parallel fields along  $r$ , the  $\lambda_j$  are non-negative (uniform) constants and  $u = \sum_j a_j E_j(0)$ .

LEMMA 3.2. *Let  $s \geq 0$ . The second fundamental forms of every boundary polyhedron  $V_E^{n-k}(s)$  with respect to outer angles in  $V(s)$  are uniformly bounded by a constant independent of  $E, k$  and  $s$ .*

*Proof.* Since the claim is local we can work in the universal covering space  $X$ . As we noted above the preimage of  $V_E^{n-k}(s)$  in  $X$  under the projection  $\pi$  is the union of a finite number of horospherical sets

$$\mathcal{H}_{i\Theta}(s) = \bigcap_{\alpha \in \Theta} \mathcal{H}_{i\alpha}(s) \subset \bigcap_{\alpha \in \Theta} \{\tau_\alpha^{-1} \tilde{h}_{i\alpha} = -s\},$$

where  $\Theta$  is a subset of  $\Delta$  with  $k$  elements. The (inner) unit normal field of the horosphere  $\{\tau_\alpha^{-1} \tilde{h}_{i\alpha} = -s\}$  is given by  $Z_{i\alpha} := -\text{grad } \tilde{h}_{i\alpha}$  (see e.g. [HI] Proposition 3.1). Using  $d\pi$  any element in the outer angle  $O(\pi(p))$  of  $V_E^{n-k}(s)$  at a point  $\pi(p) \in V_E^{n-k}(s)$  can then be identified with a positive linear combination (of norm 1) of the radial fields  $Z_{i\alpha}(p)$ ,  $\alpha \in \Theta$ . It therefore suffices to show that for any pair  $(i, \alpha)$  the second fundamental form of  $V_E^{n-k}(s)$  relative to  $d\pi Z_{i\alpha}$  is uniformly bounded. We fix  $i$  and  $\alpha$  and write  $Z$  for  $Z_{i\alpha}$ . For  $p \in X$  let  $\langle \cdot, \cdot \rangle_p$  denote the Riemannian metric of  $X$  at  $p$ . Let  $u, v \in T_p X$  be such that  $d\pi(u), d\pi(v) \in T_{\pi(p)} V_E^{n-k}(s)$ . Using the above identifications the second fundamental form of  $V_E^{n-k}(s) \subset V(s)$  with respect to  $Z$  can be written as

$$\Pi_Z(u, v)(p) = \langle D_u Z, v \rangle_p.$$

According to [HI], Proposition 3.1, we have  $D_u Z(p) = J'_u(0)$  where  $J_u$  is the stable Jacobi field along the (unique) geodesic ray, say  $r$ , in  $X$  which joins  $p$  to  $c_{i\alpha}(\infty) \in \partial_\infty X$  and with initial value  $J_u(0) = u$ . By Lemma 3.1 there are orthonormal parallel fields  $E_j(s)$  along  $r$  and constants  $\lambda_j \geq 0$  such that  $J_u(s) = \sum_j e^{-\lambda_j s} a_j E_j(s)$  with  $u = \sum_j a_j E_j(0)$ . Consequently we get  $J'_u(0) = -\sum_j \lambda_j a_j E_j(0)$  and finally, for  $v = \sum_j b_j E_j(0)$ ,  $|\Pi_Z(u, v)(p)| = |-\sum_j \lambda_j a_j b_j| \prec \|u\| \|v\|$ .  $\square$

We next estimate the volumes of the boundary polyhedra. Recall from Section 2.1 the parametrization of  $X$  by horocyclic coordinates

$$\mu: Y = U \times Z \times A \longmapsto X; (u, \tau(m), a) \longmapsto uma \cdot x_0.$$

Let  $dx^2$  be the  $G$ -invariant Riemannian metric on  $X$  induced by the Cartan-Killing form of the Lie algebra  $\mathfrak{g}$  of  $G$  and let  $dz^2$  be the invariant metric on  $Z$ . Further let  $da^2$  (resp.  $du^2$ ) be the left-invariant metric on  $A$  (resp.  $U$ ). Finally set  $dy^2 := \mu^* dx^2$ .

LEMMA 3.3. *Let  $dv_Y$ ,  $dv_U$ ,  $dv_Z$  and  $dv_A$  denote the volume elements of the metrics  $dy^2$ ,  $du^2$ ,  $dz^2$  and  $da^2$ . Then at the point  $(u, z, a) \in Y$  we have*

$$2^e dv_Y = \rho(a)^{-1} dv_U \wedge dv_Z \wedge dv_A$$

where  $e = \frac{1}{2} \dim U$  and  $\rho$  is the sum of all positive roots (counted with multiplicity); it can be written in the form  $\rho = \sum_{\alpha \in \Delta} c_\alpha \alpha$ ,  $c_\alpha > 0$ .

For the proof see [B2] Corollary 4.4.

LEMMA 3.4. *For the  $(n - k)$ -dimensional volume of each boundary polyhedron  $V_E^{n-k}(s)$  of  $V(s)$  one has the estimate*

$$\text{Vol}(V_E^{n-k}(s)) \prec s^{q-k} e^{-cs},$$

where  $q = \dim A$  is the  $\mathbb{Q}$ -rank of  $\mathbf{G}$  and  $c > 0$  is a constant (independent of  $E, k$  and  $s$ ).

*Proof.* We again consider the preimage of  $V_E^{n-k}(s)$  in  $X$  under the map  $\pi$ . We need to estimate the volume of each horospherical piece

$$\mathcal{H}_{i\Theta}(s) = \bigcap_{\alpha \in \Theta} \{\tau_\alpha^{-1} \tilde{h}_{i\alpha} = -s\} \cap \mathcal{S}_i(s), \quad i \in E.$$

It clearly suffices to carry out the estimates for  $i = 1$ ; note that  $q_1 = e$ . For the horocyclic coordinate map  $\mu: Y \rightarrow X$  and the canonical projection



$\pi_A : Y \rightarrow A$  we set  $A_\Theta(s) := \pi_A \circ \mu^{-1}(\mathcal{H}_{1\Theta}(s)) \subset A$ . The set  $A_\Theta(s)$  is contained in an “affine” subspace of  $A$  of the form  $a_1 a_*(s) A^{q-k}$  where  $a_1 a_*(s) \in A$  and  $A^{q-k}$  is a  $q-k$ -dimensional subgroup of  $A$  (see Sections 3 and 4 of [L2]). We denote the restriction of  $dv_A$  to  $A^{q-k}$  by  $dv_{A^{q-k}}$ ; for  $k = q$  we have  $A^0 = e$  and we set  $dv_{A^0} \equiv 1$ . By Lemma 3.3 we have (for  $k$  equal to the number of elements of  $\Theta$ )

$$\text{Vol}(V_E^{n-k}(s)) \prec \int_{\mu^{-1}(\mathcal{H}_{1\Theta}(s))} \rho(a)^{-1} dv_U \wedge dv_Z \wedge dv_{A^{q-k}}.$$

Since the horospherical piece  $\mathcal{H}_{1\Theta}(s)$  is part of a Siegel set  $\mathcal{S}_{\omega, \tau}$  with  $\omega$  relatively compact (and hence of finite volume) in  $UM$  we get

$$\begin{aligned} \int_{\mu^{-1}(\mathcal{H}_{1\Theta}(s))} \rho(a)^{-1} dv_U \wedge dv_Z \wedge dv_{A^{q-k}} &\prec \\ &\prec \int_{\omega} dv_U \wedge dv_Z \int_{A_\Theta(s)} \rho(a)^{-1} dv_{A^{q-k}} \prec \int_{A_\Theta(s)} \rho(a)^{-1} dv_{A^{q-k}}. \end{aligned}$$

Also by definition of a Siegel set we have  $\alpha(a) \geq \tau \succ 1$  for all  $\alpha \in \Delta$ . Moreover, the computations in the proof of Lemma 4.1 (and Lemma 3.5) in [L2] show that for all  $\alpha \in \Theta$  one has  $\alpha(a_1 a_*(s)) \succ e^{\mu_\alpha s}$  with  $\mu_\alpha > 0$ . Hence, as  $\Theta \subset \Delta$  is not empty and since  $\rho = \sum_{\alpha \in \Delta} c_\alpha \alpha$  ( $c_\alpha > 0$ ), there is a uniform constant  $c > 0$  such that  $\rho(a)^{-1} \prec e^{-cs}$  for all  $a \in A_\Theta(s)$ . As noted above the set  $A_\Theta(s)$  is contained in a  $(q-k)$ -dimensional affine cone in  $A$ . It is similar (in the sense of Euclidean geometry) to  $A_\Theta(0)$  with similarity factor  $s$  (see the proof of Lemma 4.1 in [L2]). Hence we eventually get  $\int_{A_\Theta(s)} dv_{A^{q-k}} \prec s^{q-k}$  and the Lemma follows.  $\square$

#### 4. A NEW PROOF OF THE GAUSS-BONNET FORMULA

In this section we present a new simplified proof of the Gauss-Bonnet theorem for higher rank locally symmetric spaces.

**THEOREM 4.1.** *Let  $X$  be a Riemannian symmetric space of noncompact type and  $\mathbb{R}$ -rank  $\geq 2$  and let  $\Gamma$  be an irreducible, torsion-free (non-uniform) lattice in the group of isometries of  $X$ . Then for the locally symmetric space  $V = \Gamma \backslash X$  the Gauss-Bonnet formula holds:*

$$\chi(V) = \int_V \Psi dv.$$

*Proof.* By Proposition 2.2 there is an exhaustion  $V = \bigcup_{s \geq 0} V(s)$  of  $V$  by Riemannian polyhedra  $V(s)$ . Each polyhedron  $V(s)$  in this exhaustion is equipped with the Riemannian metric induced by the one of  $V$ . Proposition 1.1 applied to  $V(s)$  yields

$$\left| (-1)^n \chi'(V(s)) - \int_{V(s)} \Psi dv \right| \prec \sum_{k=1}^q \sum_E \int_{V_E^{n-k}(s)} \int_{O(p)} \|\Psi_{E,k}\| d\omega_{k-1} dv_E(p)$$

where  $q = \dim A$  is the  $\mathbb{Q}$ -rank of  $\mathbf{G}$  (see Section 2.1) and where the index  $E$  runs through a finite set. As we remarked in Section 1 the function  $\Psi_{E,k}$  is locally computable from the components of the metric and the curvature tensor of  $V(s)$  and from the components of the second fundamental form of  $V_E^{n-k}(s)$  in  $V(s)$ . The fact that  $V$  is locally symmetric together with Lemma 3.2 thus implies that  $\|\Psi_{E,k}\| \prec 1$  for all  $E, k$ . Using Lemma 3.4 we conclude that

$$\left| (-1)^n \chi'(V(s)) - \int_{V(s)} \Psi dv \right| \prec \sum_{k,E} \text{Vol}(V_E^{n-k}(s)) \prec e^{-cs} \sum_{k=1}^q s^{q-k}.$$

By Proposition 2.3 we have  $\chi'(V(s)) = \chi(V)$ . The polyhedra  $V(s)$  exhaust  $V$  and  $\chi(V)$  is an integer; hence  $(-1)^n \chi(V) = \int_{V(s)} \Psi dv$  for sufficiently large  $s$ . Finally, for  $n$  odd  $\Psi \equiv 0$  by definition (see [AW]) and the claimed formula follows.  $\square$

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