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Autor: Tenenbaum, Gérald

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UNIFORM DISTRIBUTION ON DIVISORS AND BEHREND SEQUENCES

by Gérald TENENBAUM

1. DEFINITIONS AND BASIC RESULTS

The purpose of this paper is twofold: to give a consistent, largely self contained account on the theory of uniform distribution on divisors, and to establish effective estimates with immediate applications to the construction of Behrend sequences.

We recall that a strictly increasing sequence $\mathscr{A}$ of integers exceeding 1 is called a Behrend sequence if its set of multiples

$$\mathscr{M}(\mathscr{A}) := \{ma : a \in \mathscr{A}, m \geqslant 1\}$$

has asymptotic density 1. As underlined by Erdős in [5], the problem of characterising Behrend sequences appears to be both very difficult and fundamental for describing the multiplicative structure of normal integers. Recent progress in the area of sets of multiples and Behrend sequences may be found in [6], [12], [15], [22], [24].

The definition of uniform distribution on divisors is due to Hall [9]. It may certainly be regarded as a concept of independent interest, which is worth being developed for its own sake. The idea is to give a rigourous content, given an arithmetic function f, to the assertion that, for almost all integers n, the numbers $f(d)$ are evenly distributed modulo 1 when d runs through the divisors of n. To this end, we define the *discrepancy* function

$$\Delta(n;f) := \sup_{0 \leqslant u \leqslant v \leqslant 1} \left| \sum_{d \mid n,\, u < \langle f(d) \rangle \leqslant v} 1 - (v-u)\tau(n) \right| ,$$

where, here and throughout this paper, we let $\langle u \rangle$ denote the fractional part of the real number u. We then say that f *is uniformly distributed on divisors* (in short: erd, for the French *équirépartie sur les diviseurs*) if

(1) $$\Delta(n;f) = o(\tau(n)) \quad \text{pp},$$

where $\tau(n)$ stands for the number of divisors of n. Here and in the sequel we use the notation pp (resp. pp$l$) to indicate that a relation holds on a sequence of asymptotic density 1 (resp. logarithmic density 1).

In 1978, Hall [10] introduced the closely connected notion of divisor density. An integer sequence $\mathscr{A}$ is said to have divisor density z, in which case we write $\mathrm{D}\mathscr{A} = z$, if

$$\tau(n, \mathscr{A}) := \sum_{d \mid n,\, d \in \mathscr{A}} 1 = \{z + o(1)\}\tau(n) \qquad \text{pp}.$$

The link with uniform distribution on divisors is as follows. Writing

$$\mathscr{A}(z; f) = \{d \geqslant 1 : \langle f(d) \rangle \leqslant z\}, \tag{2}$$

we obviously have

$$\mathrm{D}\mathscr{A}(z; f) = z \qquad (z \in [0, 1]) \tag{3}$$

whenever f is erd. Moreover, as one might expect from classical results in the theory of uniform distribution modulo 1, it is not very difficult to prove that this last condition is also sufficient.

THEOREM 1 (Hall [11]). *Let* f *be an arithmetic function. Then* f *is* erd *if, and only if, condition (3) holds.*

Proof. We only need to show that the condition is sufficient. Suppose that f is not erd. Then, for suitable $\varepsilon > 0$, we have $\Delta(n; f) > 4\varepsilon\tau(n)$ for all integers n in a sequence $\mathscr{B}$ with positive lower density. Hence for each $n \in \mathscr{B}$ there exists $z_n \in [0, 1]$ such that $|T_f(n, z_n) - z_n\tau(n)| > 2\varepsilon\tau(n)$, with

$$T_f(n, z) := \tau(n, \mathscr{A}(z; f)) = |\{d \mid n : \langle f(d) \rangle \leqslant z\}|. \tag{4}$$

Let q be any integer $> 1/\varepsilon$. By the monotonicity of the function $z \mapsto T_f(n, z)$, we can find an integer a, $0 \leqslant a \leqslant q$, such that

$$|T_f(n, a/q) - (a/q)\tau(n)| > \varepsilon\tau(n).$$

Since $\mathscr{B}$ has positive lower density, this implies that (3) cannot hold for $z = a/q$.

Davenport & Erdős [2], [3], proved that a set of multiples necessarily has logarithmic density, equal to its lower asymptotic density. This implies that a (necessary and) sufficient condition for an integer sequence $\mathscr{A}$ to be Behrend is that $\delta\mathscr{M}(A) = 1$. Here and in the remainder of the paper, we use the letter δ to denote logarithmic density. The following result is a criterion for divisor density much in the same spirit.

THEOREM 2 (Hall & Tenenbaum [13]). *Let* $\mathscr{A}$ *be an integer sequence. Then we have* $\mathrm{D}\mathscr{A} = z$ *if, and only if,*

$$\tau(n, \mathscr{A}) := \sum_{d \mid n,\, d \in \mathscr{A}} 1 = \{z + o(1)\}\tau(n) \quad \mathrm{pp}l.$$

The proof rests upon the Hardy-Littlewood-Karamata Tauberian theorem. An immediate corollary of Theorems 1 & 2 is the following slightly surprising statement.

COROLLARY 1. *Let* f *be an arithmetic function such that*

$$\Delta(n; f) = o(\tau(n)) \quad \mathrm{pp}l. \tag{5}$$

Then f *is* erd.

Proof. From (5), it is clear that $\tau(n, \mathscr{A}(z; f)) = \{z + o(1)\}\tau(n)\,\mathrm{pp}l$ for all $z \in [0, 1]$. By Theorem 2, it follows that (3) holds, so Theorem 1 yields the required conclusion.

Corollary 1 opens new possibilities for constructing 'thin' Behrend sequences inasmuch as pp$l$ upper bounds for the discrepancy are usually much easier to achieve than bounds valid on a set of asymptotic density 1. For convenience of further reference, we make a formal statement.

THEOREM 3. *Let* $\varepsilon(n)$ *be a non-increasing function of* n *such that*

$$\varepsilon(n) = o(1), \qquad \varepsilon(n)\tau(n) \to \infty \qquad \mathrm{pp}l,$$

and let f *be an arithmetic function satisfying*

$$\Delta(n; f) < \tfrac{1}{2}\varepsilon(n)\tau(n) \quad \mathrm{pp}l.$$

Then the integer sequence

$$\mathscr{A} = \{d \geqslant 1 : \langle f(d) \rangle \leqslant \varepsilon(d)\}$$

is a Behrend sequence.

Proof. We plainly have

$$|\{d \mid n : \langle f(d) \rangle \leqslant \varepsilon(n)\}| \geqslant \varepsilon(n)\tau(n) - \Delta(n; f) > \tfrac{1}{2}\varepsilon(n)\tau(n) \qquad \mathrm{pp}l.$$

Since $\varepsilon(d) \geqslant \varepsilon(n)$ whenever $d \mid n$, this implies $\delta\mathscr{M}(\mathscr{A}) = 1$. By the Davenport-Erdős theorem, we deduce that $\mathscr{A}$ is a Behrend sequence.

Thus the problem of finding effective bounds on a set of logarithmic density 1 appears to be essential in both problems of obtaining erd-type results and of constructing Behrend sequences. The remainder of this paper is devoted to describing the various methods that have been devised to this end.

An obvious consequence of Theorem 1 is that any effective criterion for divisor density may be employed to decide whether a given function is erd. We now quote a result of this kind. The statement involves a function $R(x) \leqslant x$ which is increasing and has the property that, for all $y \in [0, 1]$ (but actually $y = \frac{1}{4}$ suffices) there is a suitable Stieltjes measure $d\lambda_y(t)$ on $\left[0, \frac{1}{2}\right]$ with $|d\lambda_y(t)| \ll t^{-y} dt$ and

$$\sum_{n \leqslant x} y^{\Omega(n)} = \int_0^{1/2} x^{1-t} d\lambda_y(t) + O(x/R(x)) , \tag{6}$$

where, here and in the sequel, we let $\Omega(n)$ (resp. $\omega(n)$) denote the total number of prime factors of n, counted with (resp. without) multiplicity. It is shown in [13] that

$$R(x) = \exp\{(\log x)^{3/5-\varepsilon}\}$$

is an admissible choice for all $\varepsilon > 0$, and an examination of the proof shows that $x/R(x)$ is essentially of the size of the error term in the prime number theorem — see also [25], chapters II.5, II.6 and notes on §II.5.4.

THEOREM 4 (Hall & Tenenbaum [13]). *Let* $\{u_j\}_{j=0}^{\infty}$ *be a strictly increasing sequence of positive real numbers such that* $|\{j : u_j \leqslant x\}| \leqslant R(x^{o(1)})$ *and put* $\mathscr{A} := \cup_{j=1}^{\infty} (u_{2j}, u_{2j+1}] \cap \mathbf{Z}^+$. *Then* $\delta\mathscr{A} = z$ *implies that* $\mathrm{D}\mathscr{A} = z$.

Theorem 4 provides a ready-to-use sufficient condition for smooth functions to be erd. For instance, it enables one to recover immediately the two following basic results. We let $\log_k$ denote the k-fold iterated logarithm.

COROLLARY 2 (Tenenbaum [23]). *The function* $d \mapsto (\log d)^{\alpha}$ *is* erd *if, and only if,* $\alpha > 0$.

This result was conjectured by Hall in [9].

COROLLARY 3 (Hall [9], [10]). *The function* $d \mapsto (\log_2 d)^{\beta}$ *is* erd *if, and only if,* $\beta > 1$.

It is straightforward to check that the sequences $\mathscr{A}(z; f)$ defined by (2) for $f = \log^{\alpha}$ and $f = \log_2^{\beta}$ satisfy the hypotheses of Theorem 4 whenever $\alpha > 0$, $\beta > 1$. On the other hand, as observed by Hall in [9], relation (1) does not hold for $f(d) = (\log_2 d)^{\beta}$ when $\beta \leqslant 1$. Indeed, let $0 < \delta < 2^{-1/\beta}$

and consider the set $\mathscr{S}(\delta)$ of integers of the form $n = mp$ with $p^{\delta} > m$, which has natural density $\log(1+\delta)$. For $n \in \mathscr{S}(\delta)$, there are at least $\frac{1}{2}\tau(n)$ divisors d of n which are divisible by p, and all of these verify $f(p) \leqslant f(d) \leqslant f(p^{1+\delta}) < f(p) + \delta^{\beta}$. Thus $\Delta(n;f) \geqslant \left(\frac{1}{2} - \delta^{\beta}\right)\tau(n)$.

The strongest limitation in Theorem 4 is the growth condition on the sequence $\{u_j\}_{j=0}^{\infty}$, which, in the present state of knowledge concerning the error term of the prime number theorem or the zero-free region for the Riemann zeta function, certainly implies that

$$|\{j : u_j \leqslant x\}| = \exp\{o((\log x)^{3/5}(\log_2 x)^{-1/5})\}\,. \tag{7}$$

Thus, we can only obtain from Theorem 4 that

$$f(d) = \exp\{(\log d)^{\alpha}\}$$

is erd for $0 < \alpha < 3/5$, although it is natural to conjecture that this holds for all positive $\alpha \neq 1$. We shall see in section 3 that this can indeed be established for the range $0 < \alpha < 3/2$, $\alpha \neq 1$. To tackle functions f beyond the scope of Theorem 4, one possibility is to appeal, as already done in [13], to the criterion for uniform distribution on divisors established in [23]. In the spirit of the Weyl criterion for ordinary uniform distribution modulo 1, this is formulated in terms of exponential sums. We now provide an effective form of this criterion. Given an integer ν, we put

$$\varepsilon_{\nu}(x;f) := (\log x)^{-1/2} \sum_{k \leqslant x} \Bigg| \sum_{\substack{n \leqslant x \\ n \equiv 0 \,(\mathrm{mod}\, k)}} \frac{e(\nu f(n))}{n 4^{\Omega(n)}} \Bigg|\,. \tag{8}$$

Here and throughout the paper we use the traditional notation

$$e(u) = e^{2\pi i u} \, (u \in \mathbf{R})\,.$$

THEOREM 5. *Let f be an arithmetical function. Then f is* erd *if, and only if, we have, as $x \to \infty$,*

$$\varepsilon_{\nu}(x;f) = o(1) \qquad (\nu \neq 0)\,. \tag{9}$$

Furthermore, if this is the case, then the upper bound

$$\Delta(n;f) < \xi(n)\tau(n) \min_{T \geqslant 1} \left\{ \frac{1}{T^2} + \log T \sum_{1 \leqslant \nu \leqslant T} \frac{\varepsilon_{\nu}^{+}(n;f)}{\nu} \right\}^{1/2} \qquad \mathrm{pp}l \tag{10}$$

holds for arbitrary $\xi(n) \to \infty$, where $\varepsilon_{\nu}^{+}(x;f)$ is, for each ν, any non-increasing function such that $x \mapsto \varepsilon_{\nu}^{+}(x;f)\sqrt{\log x}$ is non-decreasing and $\varepsilon_{\nu}(x;f) \leqslant \varepsilon_{\nu}^{+}(x;f)$ holds for large x.

Thus, the problem of finding effective pp$l$ bounds for the discrepancy (with the byproduct, which is essential here, of exhibiting new types of Behrend sequences) may be reduced to the study of appropriate exponential sums with multiplicative coefficients.

Proof of Theorem 5. First assume that f is erd. For $k \geqslant 1$, $x \geqslant 2$, $0 \leqslant z \leqslant 1$, put

$$\Phi_k(z;x) = (\log x)^{-1/2} \sum_{\substack{n \leqslant x,\, n \equiv 0 \,(\mathrm{mod}\, k) \\ \langle f(n) \rangle \leqslant z}} \frac{1}{n4^{\Omega(n)}} \ll (\log x)^{-1/4} \frac{1}{k4^{\Omega(k)}} .$$

By the author's criterion [23] for divisor density and Theorem 1, we have that

$$F_x(z) := \sum_{k \leqslant x} \left| \Phi_k(z;x) - z\Phi_k(1;x) \right| = o(1) \quad (x \to \infty) \tag{11}$$

for all fixed $z \in [0, 1]$. Now for any non-zero integer ν we have

$$\begin{aligned} \varepsilon_\nu(x;f) &= \sum_{k \leqslant x} \left| \int_0^1 e(\nu z)\, \mathrm{d}\Phi_k(z;x) \right| \\ &= \sum_{k \leqslant x} \left| 2\pi\nu i \int_0^1 e(\nu z)\{\Phi_k(z;x) - z\Phi_k(1;x)\}\, \mathrm{d}z \right| \\ &\leqslant 2\pi |\nu| \int_0^1 F_x(z)\, \mathrm{d}z . \end{aligned}$$

Since $F_x(z) = O(1)$ uniformly in x, z, the required conclusion (9) follows by Lebesgue's theorem of dominated convergence.

Conversely, we now assume that (9) holds and derive a pp$l$ upper bound for $\Delta(n;f)$. By the Erdős-Turán inequality for the discrepancy (see e.g. Kuipers & Niederreiter [19], theorem 2.5) we have for all n and $T \geqslant 1$

$$\Delta(n;f) \ll \frac{\tau(n)}{T} + \sum_{1 \leqslant \nu \leqslant T} \frac{|g_\nu(n)|}{\nu}, \tag{12}$$

with $g_\nu(n) := \sum_{d|n} e(\nu f(d))$. By the Cauchy-Schwarz inequality, we infer that

$$\Delta(n;f)^2 \ll \frac{\tau(n)^2}{T^2} + \log T \sum_{1 \leqslant \nu \leqslant T} \frac{|g_\nu(n)|^2}{\nu} . \tag{13}$$

We now estimate $|g_\nu(n)|^2$ on logarithmic average with weight $1/4^{\Omega(n)}$. The procedure is similar to the proof of theorem 1 of [23]. Writing $\sigma = 1/\log x$, we have

$$\begin{aligned} L_\nu(x) := \sum_{n \leqslant x} \frac{|g_\nu(n)|^2}{n4^{\Omega(n)}} &\leqslant e \sum_{n=1}^{\infty} \frac{|g_\nu(n)|^2}{n^{1+\sigma}4^{\Omega(n)}} \\ &\ll \sum_{n=1}^{\infty} \frac{1}{n^{1+\sigma}4^{\Omega(n)}} \sum_{d|n,\, t|n} e(\nu f(d) - \nu f(t)) \\ &= \zeta\left(1+\sigma, \tfrac{1}{4}\right) \sum_{d,t \geqslant 1} \frac{e(\nu f(d) - \nu f(t))}{[d,t]^{1+\sigma}4^{\Omega([d,t])}}, \end{aligned} \tag{14}$$

where we used the notation

$$\zeta(s,y) = \sum_{n=1}^{\infty} \frac{y^{\Omega(n)}}{n^s} = \prod_p (1 - yp^{-s})^{-1} \qquad (|y| < 2,\ \Re e\, s > 1)\,. \tag{15}$$

We note that

$$\begin{aligned} \zeta\left(1+\sigma, \tfrac{1}{4}\right) &= \zeta(1+\sigma)^{1/4} \prod_p \left(1 - \frac{1}{4p^{1+\sigma}}\right)^{-1} \left(1 - \frac{1}{p^{1+\sigma}}\right)^{1/4} \\ &\sim H\left(\tfrac{1}{4}\right) (\log x)^{1/4} \qquad (x \to \infty)\,, \end{aligned} \tag{16}$$

with

$$H(y) := \prod_p \left(1 - \frac{y}{p}\right)^{-1} \left(1 - \frac{1}{p}\right)^{y} \qquad (|y| < 2)\,.$$

Using the identity

$$m^{1+\sigma}4^{\Omega(m)} = \sum_{k|m} \lambda(k,\sigma) k^{1+\sigma}4^{\Omega(k)}, \text{ with } \lambda(k,\sigma) := \prod_{p|k} \left(1 - \frac{1}{4p^{1+\sigma}}\right) \leqslant 1,$$

we may rewrite the last double sum in (14) as

$$\begin{aligned} &\sum_{d,t \geqslant 1} \frac{e(\nu f(d) - \nu f(t))}{(dt)^{1+\sigma}4^{\Omega(dt)}} \sum_{k|(d,t)} \lambda(k,\sigma) k^{1+\sigma}4^{\Omega(k)} \\ &= \sum_{k=1}^{\infty} \frac{\lambda(k,\sigma)}{k^{1+\sigma}4^{\Omega(k)}} \left| \sum_{m=1}^{\infty} \frac{e(\nu f(km))}{m^{1+\sigma}4^{\Omega(m)}} \right|^2. \end{aligned} \tag{17}$$

Bounding $\lambda(k,\sigma)$ by 1, and noticing that the m-sum is at most $\zeta\left(1+\sigma, \frac{1}{4}\right)$ in absolute value, we see that the quantity (17) does not exceed

$$\zeta\left(1+\sigma, \tfrac{1}{4}\right) \sum_{k=1}^{\infty} \frac{1}{k^{1+\sigma} 4^{\Omega(k)}} \left| \sum_{m=1}^{\infty} \frac{e(\nu f(km))}{m^{1+\sigma} 4^{\Omega(m)}} \right|$$

$$= \zeta\left(1+\sigma, \tfrac{1}{4}\right) \sum_{k=1}^{\infty} \left| \sum_{\substack{n=1 \\ k \mid n}}^{\infty} \frac{e(\nu f(n))}{n^{1+\sigma} 4^{\Omega(n)}} \right| .$$

Inserting this upper bound into (14) and appealing to (16) we obtain

$$L_\nu(x) \ll (\log x)^{1/2} \sum_{k=1}^{\infty} \left| \int_{0-}^{\infty} e^{-\sigma u} \mathrm{d} A_k(u) \right| ,$$

with $A_k(u) := \sum_{n \leqslant e^u, k \mid n} e(\nu f(n)) 4^{-\Omega(n)} n^{-1}$. Integrating by parts, it follows that

$$(18) \qquad \begin{aligned} L_\nu(x) &\ll \sigma (\log x)^{1/2} \sum_{k=1}^{\infty} \left| \int_0^{\infty} e^{-\sigma u} A_k(u) \mathrm{d} u \right| \\ &\ll (\log x)^{-1/2} \int_0^{\infty} A(u) e^{-\sigma u} \mathrm{d} u , \end{aligned}$$

with $A(u) := \sum_{k=1}^{\infty} |A_k(u)| \leqslant \varepsilon_\nu^+(e^u; f)\sqrt{u}$. The last integral may be easily estimated using the monotonicity properties of $\varepsilon_\nu^+(e^u; f)$. We have

$$\begin{aligned} &\int_0^{\infty} A(u) e^{-\sigma u} \mathrm{d} u \\ &\leqslant \int_0^{1/\sigma} \varepsilon_\nu^+(e^{1/\sigma}; f)\sqrt{1/\sigma}\, \mathrm{d} u + \int_{1/\sigma}^{\infty} \varepsilon_\nu^+(e^{1/\sigma}; f) e^{-\sigma u} \sqrt{u}\, \mathrm{d} u \\ &\ll (\log x)^{3/2} \varepsilon_\nu^+(x; f), \end{aligned}$$

and so $L_\nu(x) \ll (\log x) \varepsilon_\nu^+(x; f)$. Inserting this into (13), we deduce that, for any $T \geqslant 1$,

$$(19) \qquad (\log x)^{-1} \sum_{n \leqslant x} \frac{\Delta(n; f)^2}{n 4^{\Omega(n)}} \ll \frac{1}{T^2} + \log T \sum_{1 \leqslant \nu \leqslant T} \frac{\varepsilon_\nu^+(x; f)}{\nu} .$$

Using the inequality

$$\tau(n) \geqslant 2^{\Omega(n)} / \xi(n)^{1/3} \quad \text{pp},$$

which follows from the fact that $\Omega(n) - \omega(n)$ is bounded on average, we infer from (19) that

$$(20)\qquad \Delta(n;f) < \tau(n)\xi(n) \min_{T \geqslant 1} \left\{ \frac{1}{T^2} + \log T \sum_{1 \leqslant \nu \leqslant T} \frac{\varepsilon_\nu^+(x;f)}{\nu} \right\}^{1/2}$$

holds for all $n \leqslant x$ except those of a set $\mathscr{E}_x$ with $\sum_{n \in \mathscr{E}_x}(1/n) = o(\log x)$. The stated result follows since the quantity inside curly brackets in (20) is a non-increasing function of x for each fixed T.

It would be possible to obtain pp upper bounds for the discrepancy in Theorem 4 along the lines of the proof of Theorem 2, appealing to an effective form of the Hardy-Littlewood-Karamata Tauberian theorem. However, the resulting estimate would be much weaker than (10). Such an analysis might of course be pursued for its own sake, but is irrelevant in the present context, as we remarked earlier.

The pp$l$ upper bound (10) is by no means a unique or optimal choice. We now summarise what we believe to be the three most important variations.

It is convenient to introduce the following definition.

DEFINITION. *A function* $F: \mathbf{R}^+ \to \mathbf{R}^+$ *is called slowly increasing (resp. slowly decreasing) if it satisfies for suitable* $x_0 > 0$

$$F(x) \ll_\varepsilon F(x^\varepsilon) \quad (\text{resp. } F(x) \gg_\varepsilon F(x^\varepsilon)) \quad (x > x_0)$$

for all $\varepsilon \in]0, 1[$.

Recall the formula

$$g_\nu(n) := \sum_{d \mid n} e(\nu f(n)) .$$

Then it is an immediate consequence of the Erdős-Turán inequality (12) that

$$(21)\quad \sum_{n \leqslant x} \frac{\Delta(n;f)}{n} \left(\frac{y}{2}\right)^{\Omega(n)} \ll \frac{(\log x)^y}{T} + \sum_{1 \leqslant \nu \leqslant T} \frac{1}{\nu} \sum_{n \leqslant x} \frac{|g_\nu(n)|}{n} \left(\frac{y}{2}\right)^{\Omega(n)},$$

uniformly for $x \geqslant 2$, $T \geqslant 1$, $0 \leqslant y \leqslant y_0 < 4$. Suppose $E_1(x,y)\log x$ is an upper bound for the right-hand side of (21), corresponding to some optimal or quasi-optimal choice $T = T(x,y)$, which has the property that $x \mapsto E_1(x,y)$ is slowly increasing. Then we deduce from (21) the following statement.

THEOREM 6. *Let* $\xi(n) \to \infty$ *and* $0 < y_0 < 4$. *Then, for any function* $y = y(n)$ *with values in* $[0, y_0]$ *such that* $n \mapsto E_1(n, y(n))$ *is slowly increasing, we have*

$$(22)\qquad \Delta(n;f) < \xi(n)\tau(n)E_1(n,y)y^{-\Omega(n)} \quad \text{pp}l .$$

Of course, from our assumption on $E_1(x, y)$, any $y(n) = y$ independent of n will be an admissible choice. Since $\Omega(n)$ has normal order $\log_2 n$, the optimal function $y(n)$ in (22) will be close to $y = y_1(n)$ minimising the expression

$$(\log n)^{-\log y} E_1(n, y) \,, \tag{23}$$

and indeed this choice always approximates the minimum of the right-hand side of (22) to within a factor $(\log n)^{o(1)}$.

Theorem 6 is only applicable when one disposes of non-trivial estimates for the right-hand side of (21). This is in particular the case when f is additive, for g_ν is then multiplicative. We shall study this situation in detail in section 5.

When individual bounds for $|g_\nu(n)|$ fail to yield non-trivial information on the weighted average appearing in (21), one can still perform a computation parallel to (13)-(17), but with $\left(\frac{1}{4}\right)^{\Omega(n)}$ replaced by $\left(\frac{1}{4}y\right)^{\Omega(n)}$. This gives, uniformly for $x \geqslant 2$, $T \geqslant 1$, $0 \leqslant y \leqslant y_0 < 8$,

$$\begin{aligned}(\log x)^{-1} \sum_{n \leqslant x} \frac{\Delta(n; f)^2}{n} \left(\frac{y}{4}\right)^{\Omega(n)} \\ \ll \frac{(\log x)^{y-1}}{T^2} + \log T (\log x)^{y/4-1} \sum_{1 \leqslant \nu \leqslant T} \frac{1}{\nu} H_\nu(x, y) \,,\end{aligned} \tag{24}$$

with

$$H_\nu(x, y) := \sum_{k=1}^{\infty} \left(\frac{y}{4}\right)^{\Omega(k)} \frac{1}{k^{1+\sigma}} \left| \sum_{m=1}^{\infty} \left(\frac{y}{4}\right)^{\Omega(m)} \frac{e(\nu f(km))}{m^{1+\sigma}} \right|^2 \quad (\sigma := 1/\log x) \,. \tag{25}$$

At this stage, we may employ two distinct strategies. The first one corresponds to cases in which we can take advantage of the presence of the squared modulus in (25). If we then denote by $E_2(x, y)$ an upper bound for the right-hand side of (24) which is slowly increasing as a function of x, we obtain the following result.

THEOREM 7. *Let* $\xi(n) \to \infty$ *and* $0 < y_0 < 8$. *Then, for any function* $y = y(n)$ *with values in* $[0, y_0]$ *and such that* $n \mapsto E_2(n, y(n))$ *is slowly increasing, we have*

$$\Delta(n; f) < \xi(n) \tau(n) y^{-\Omega(n)/2} \sqrt{E_2(n, y)} \quad \text{pp}l \,. \tag{26}$$

Here again the optimal y must be close to $y = y_2(n)$ minimising the expression

$$(\log n)^{-\log y} E_2(n, y) \,.$$

The second strategy, which corresponds to cases when a 'linearized' upper bound is more convenient, consists in bounding trivially one of the two (identical) factors of the square in (25) by $\zeta(1+\sigma, y)$ and then repeating *mutatis mutandis* the procedure described in (18)-(20). For $0 < y < 8$, $\nu \geqslant 1$, let $\varepsilon_\nu^+(x, y; f)$ be a non-increasing function of x such that $x \mapsto \varepsilon_\nu^+(x, y; f)(\log x)^{y/2}$ is non-decreasing and, for all $x \geqslant 2$,

$$(27) \qquad (\log x)^{-y/2} \sum_{k \leqslant x} \Bigg| \sum_{\substack{n \leqslant x \\ n \equiv 0 \pmod k}} \left(\frac{y}{4}\right)^{\Omega(n)} \frac{e(\nu f(n))}{n} \Bigg| \leqslant \varepsilon_\nu^+(x, y; f).$$

We arrive at the following estimate generalising (19): the bound

$$(28) \qquad \sum_{n \leqslant x} \left(\frac{y}{4}\right)^{\Omega(n)} \frac{\Delta(n; f)^2}{n} \ll (\log x)^y E_3(x, y)$$

holds uniformly for $x \geqslant 2$, $0 \leqslant y \leqslant y_0$, with

$$E_3(x, y) := \min_{T \geqslant 2} \left\{ \frac{1}{T^2} + \log T \sum_{1 \leqslant \nu \leqslant T} \frac{\varepsilon_\nu^+(x, y; f)}{\nu} \right\}.$$

The monotonicity hypotheses on the functions $\varepsilon_\nu^+(x, y; f)$ are slightly awkward in practical use, and, for convenience of further reference, we note right away that they may be slightly relaxed. Let us say that a positive function F is *weakly increasing* (resp. *weakly decreasing*) if it satisfies $F(t) \ll F(x)$ (resp. $F(t) \gg F(x)$) for $t \leqslant x$. Then it is enough for (28) to assume that $\varepsilon_\nu^+(x, y; f)$ and $(\log x)^{y/2} \varepsilon_\nu^+(x, y; f)$ are respectively weakly decreasing and weakly increasing functions of x.

The upper bound (28) immediately implies, in a straightforward way, our next theorem.

THEOREM 8. *Let* $\xi(n) \to \infty$, $0 < y_0 < 8$, $y = y(n) \in [0, y_0]$ *and suppose that* $E_3^*(n, y)$ *is an upper bound for* $E_3(n, y)$ *which is slowly increasing as a function of* n. *Then we have*

$$(29) \qquad \Delta(n; f) < \xi(n) \tau(n) (\log n)^{(y-1)/2} y^{-\Omega(n)/2} \sqrt{E_3^*(n, y)} \qquad \mathrm{pp}l.$$

As before, we remark that the choice $y = y_3(n)$ minimising the expression

$$(\log n)^{y - 1 - \log y} E_3(n, y)$$

yields an approximation of the optimum to within a factor $(\log n)^{o(1)}$.